Support Vector Machines

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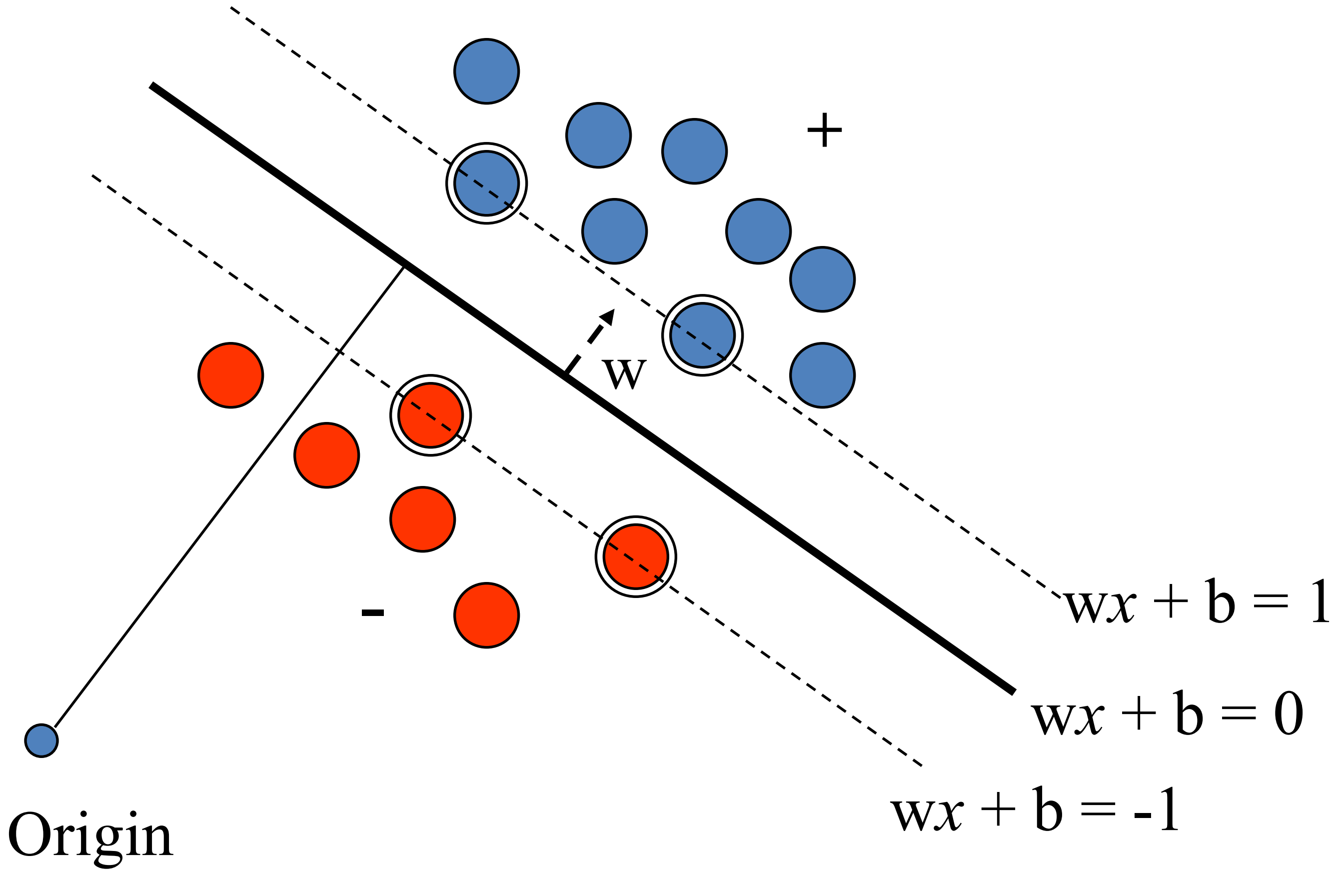
Slides Adapted from Book and CMU, Stanford Machine Learning Courses

Deterministic Learning Machine

- Learning a mapping: $x_i \mapsto y_i$.
- The machine is defined by a set of mappings (functions): $f(x, a)$
- $f(x,a)$ are defined by the adjustable parameters a. The machine is assumed to be deterministic.
- A particular choice of a generates a “trained” machine (**examples?**)

Linear Classification Hyperplane

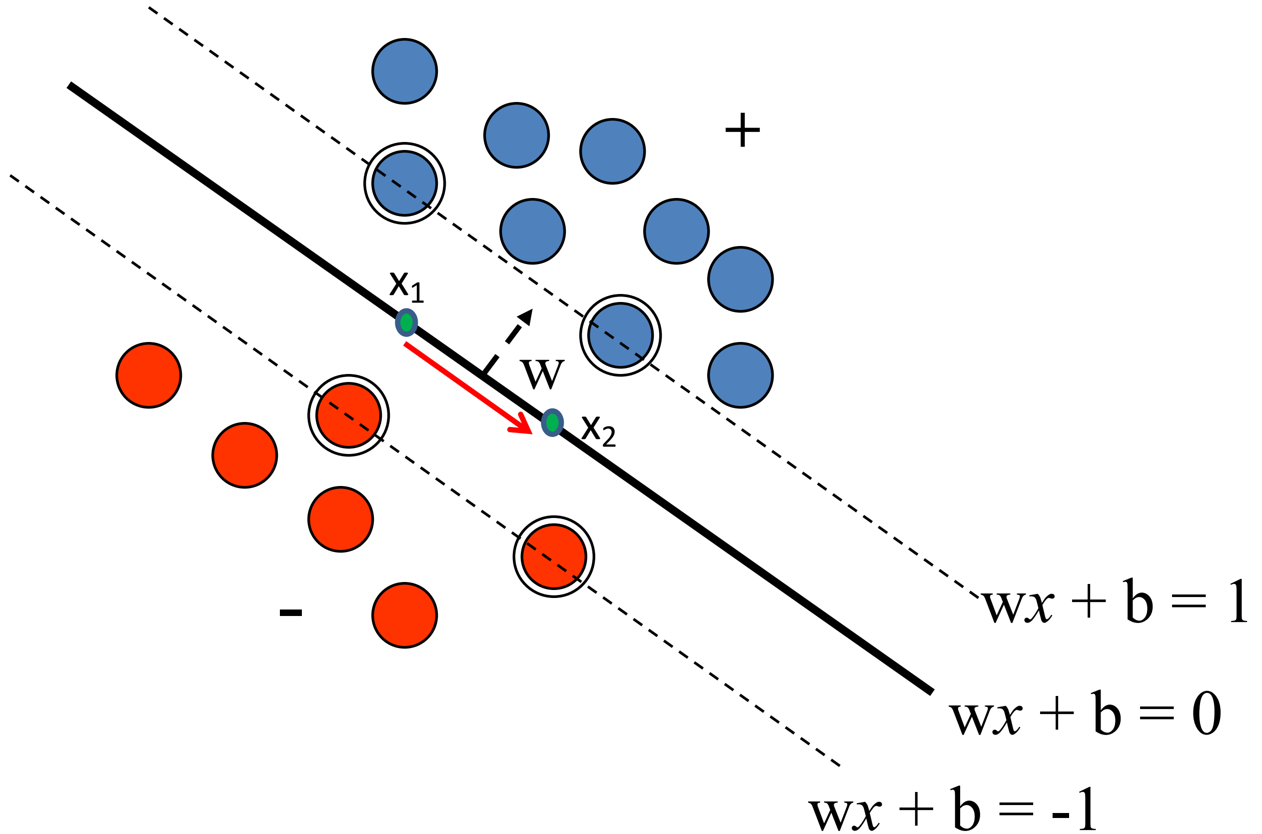
- A set of labeled training data $\{x_i, y_i\}$, i = 1, ..., l, x_i in R^d, y_i in {-1, 1}.
- A linear machine trained on the separable data.
- A linear hyperplane $f(x) = w.x+b$, separates the positive from negative examples, w is normal to the hyperplane.
- The points which lie on the hyperplane satisfy $w.x + b = 0$, positives $w.x+b > 0$, and negatives $w.x+b < 0$.



$x_i \,.\, w + b >= +1$ for $y_i = +1$
$x_i \,.\, w + b <= -1$, for $y_i = -1$

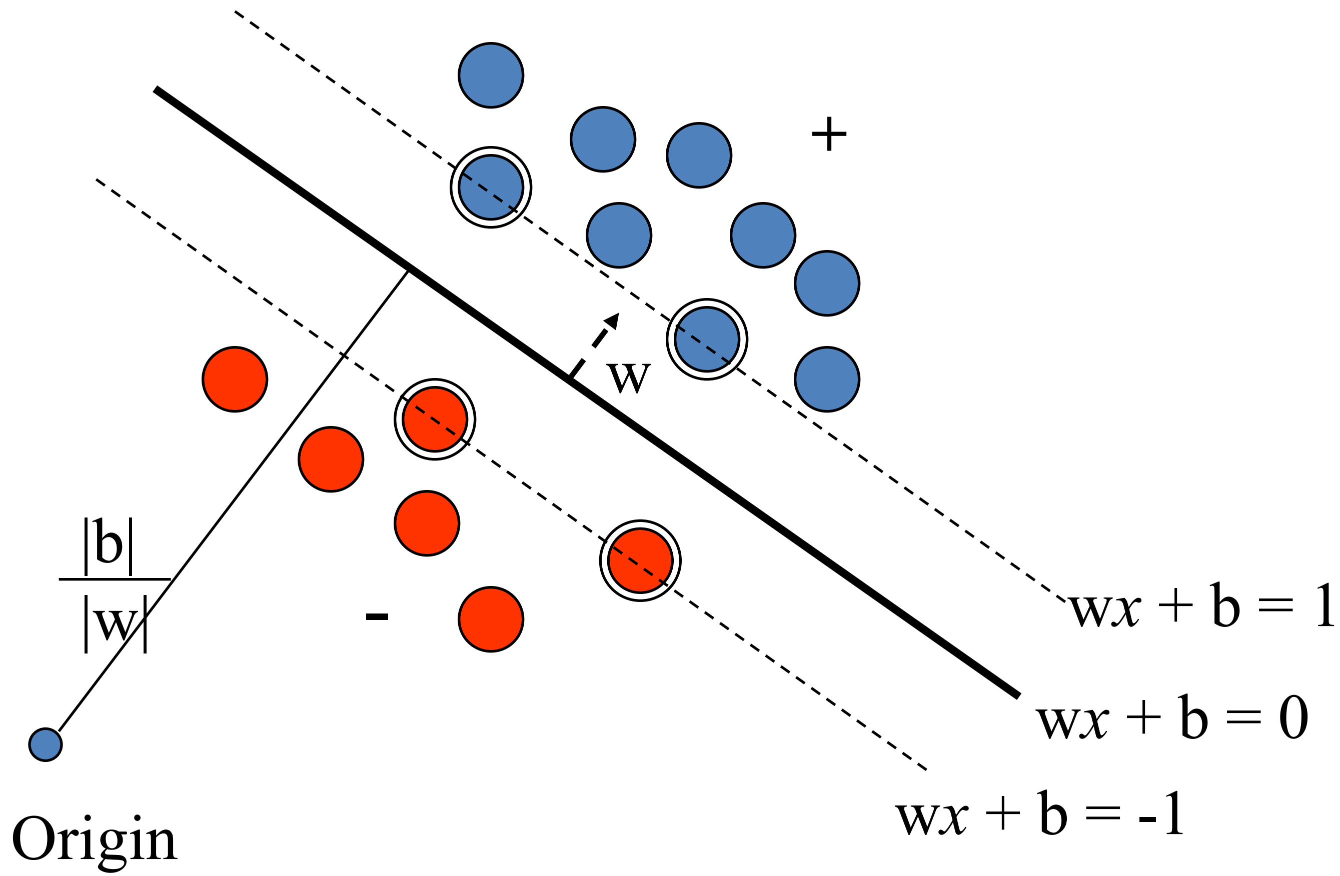
combined into: $y_i(x_i.w+b) >= 1$

Comments: equivalent to general form $y_i(x_i.w+b) >= c$



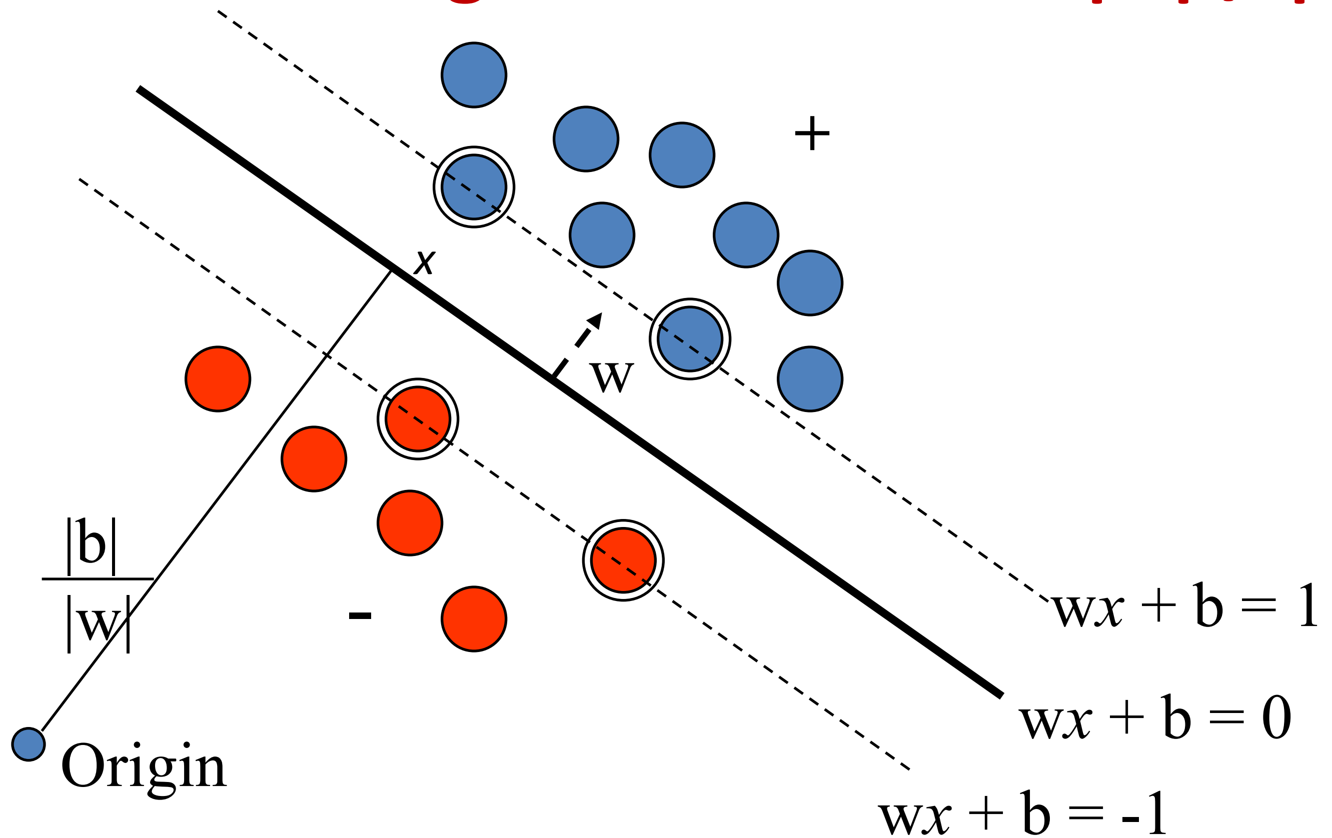
Prove w is normal (perpendicular) to the hyperplane

$$(x_2 - x_1) \cdot w = x_2 . w - x_1 . w = -b - (-b) = 0$$



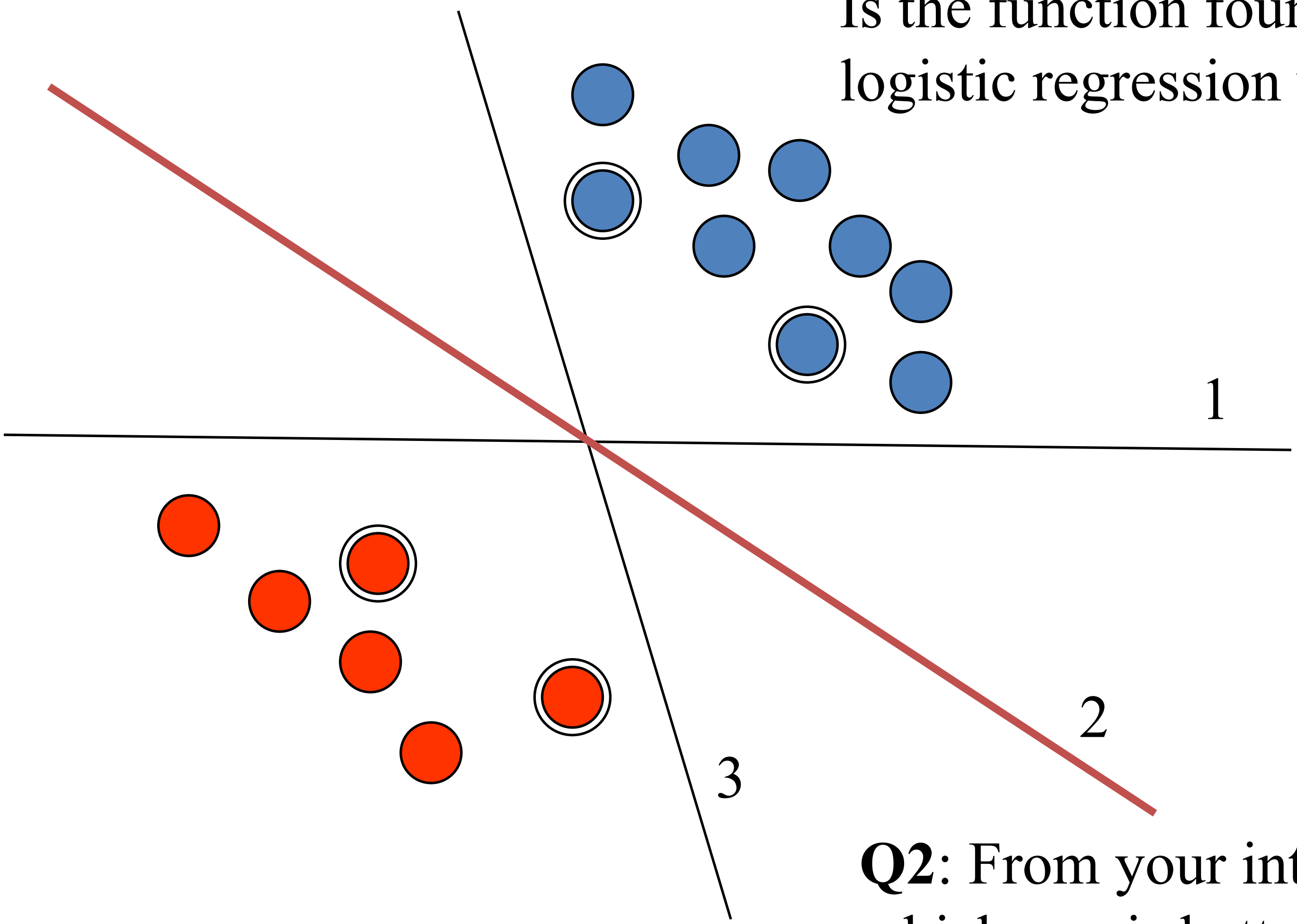
+
w
-
|b|
|w|
$wx + b = 1$
$wx + b = 0$
$wx + b = -1$
Origin

Distance from origin to wx + b=0 is |b| / |w|



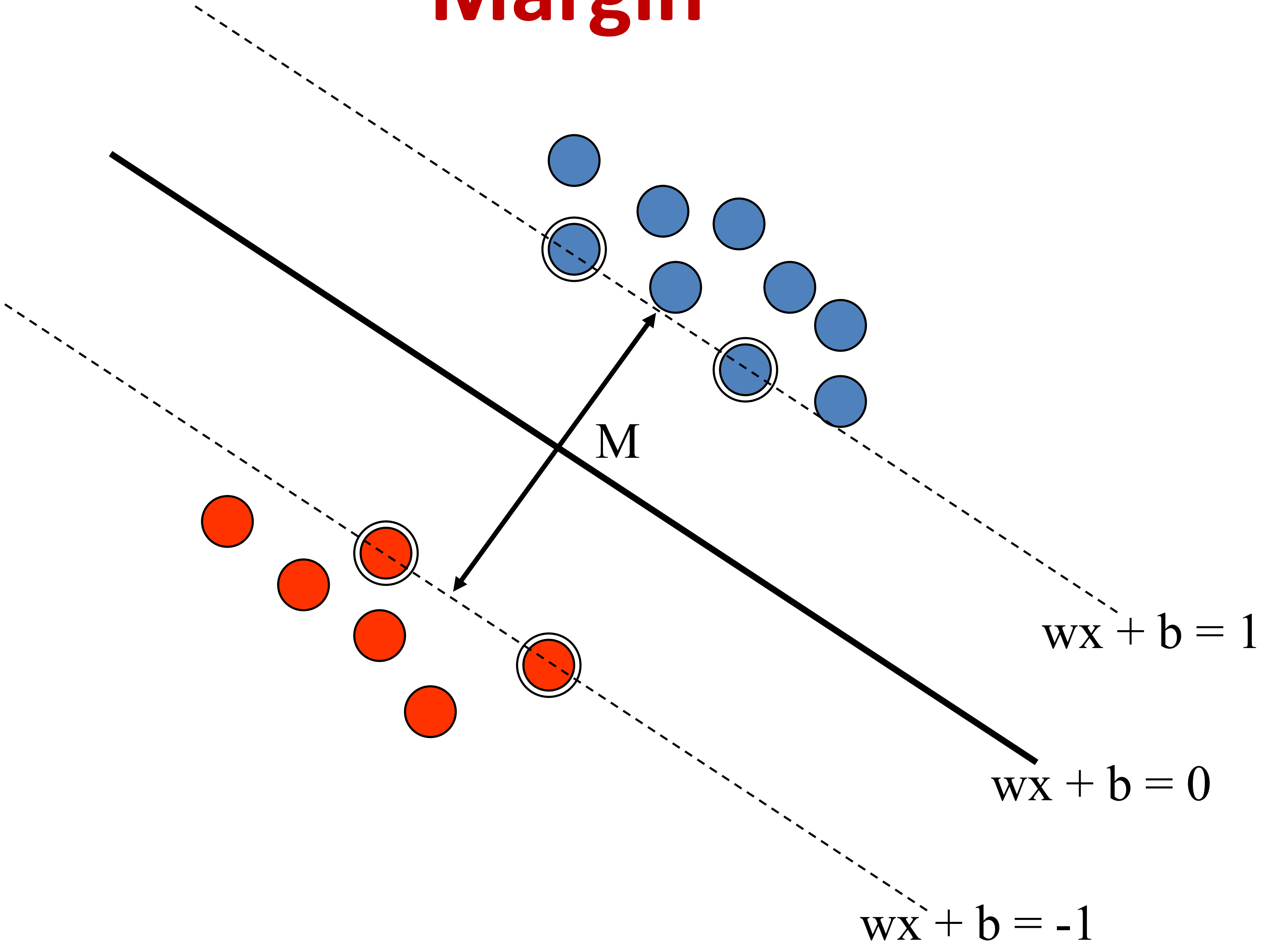
- Choose a point x on wx+b=0 such that vector (0, x) is perpendicular to wx+b = 0. So x is λw because w is norm of wx+b=0.
- So λw.w+b = 0 ➔ λ = -b/ w.w = -b / $|w|^2$
- So x = -b / $|w|^2$ *w ➔ |x| = |b| / |w|.

Q1: How logistic regression finds a linear hyperplane? Is the function found by logistic regression unique?

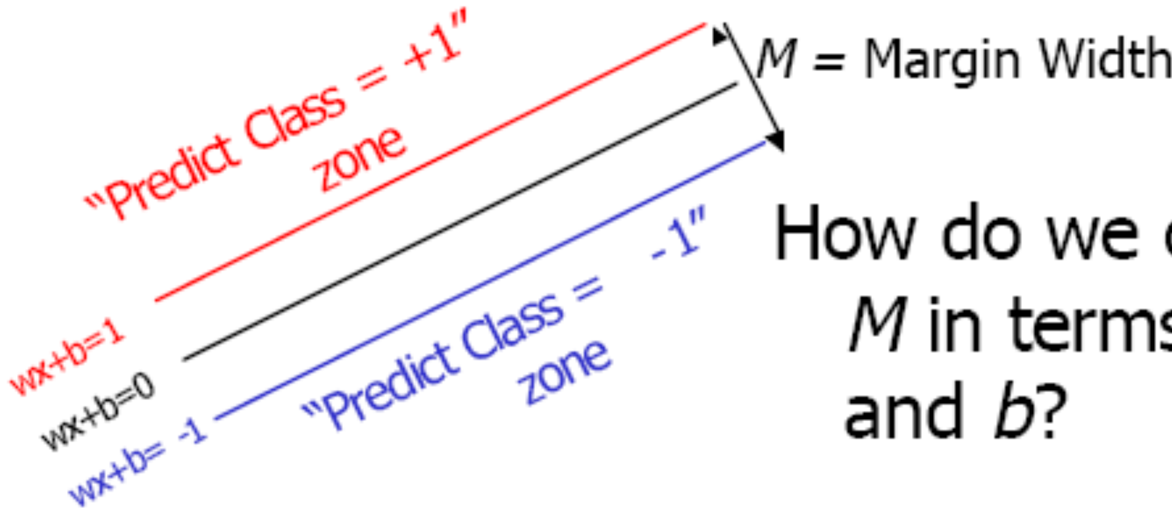


Q2: From your intuition, which one is better?

Margin



How to Compute Margin?

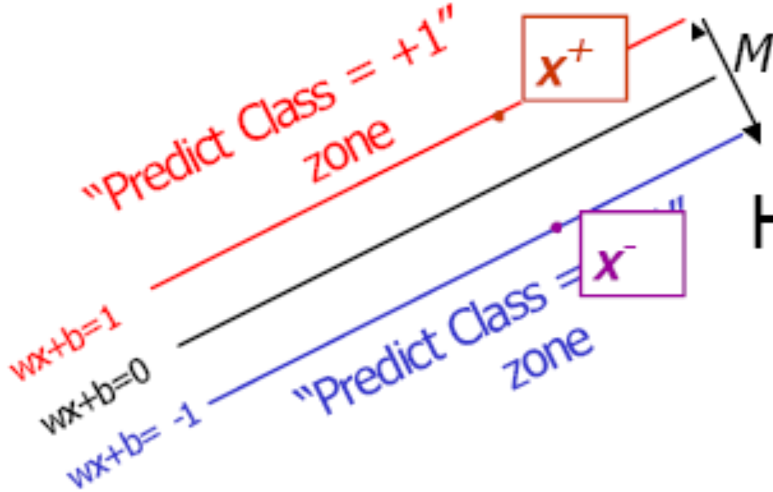


How do we compute M in terms of $\boldsymbol{w}$ and b?

- Plus-plane = $\{ \mathbf{x} : \mathbf{w} \cdot \mathbf{x} + b = +1 \}$
- Minus-plane = $\{ \mathbf{x} : \mathbf{w} \cdot \mathbf{x} + b = -1 \}$

A. Moore, 2003

Computing the margin width



How do we compute *M* in terms of **w** and *b*?

- Plus-plane = $\{ \mathbf{x} : \mathbf{w} \,.\, \mathbf{x} + b = +1 \}$
- Minus-plane = $\{ \mathbf{x} : \mathbf{w} \,.\, \mathbf{x} + b = -1 \}$
- The vector **w** is perpendicular to the Plus Plane
- Let $\mathbf{x}^-$ be any point on the minus plane
- Let $\mathbf{x}^+$ be the closest plus-plane-point to $\mathbf{x}^-$.

Any location in R^m: not necessarily a datapoint

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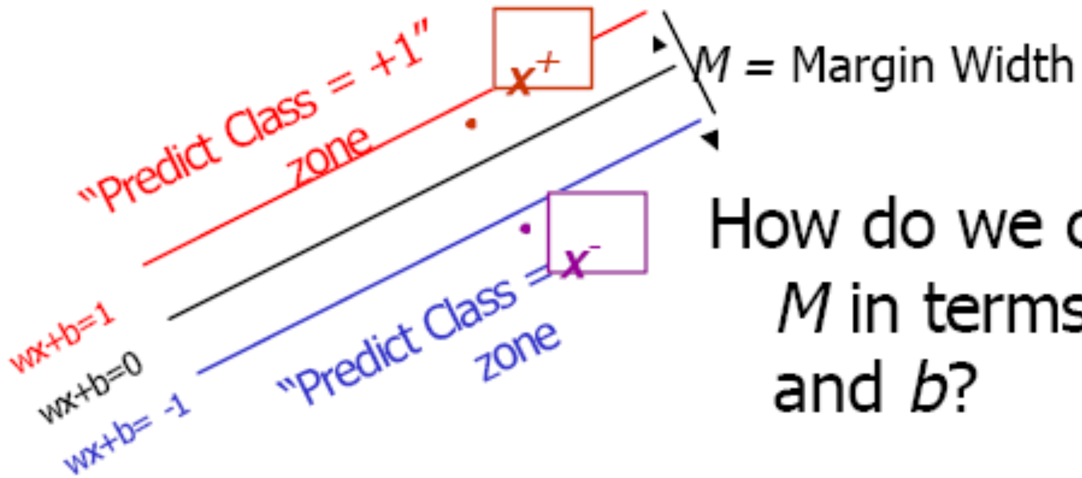
Computing the margin width



How do we compute *M* in terms of ***w*** and *b*?

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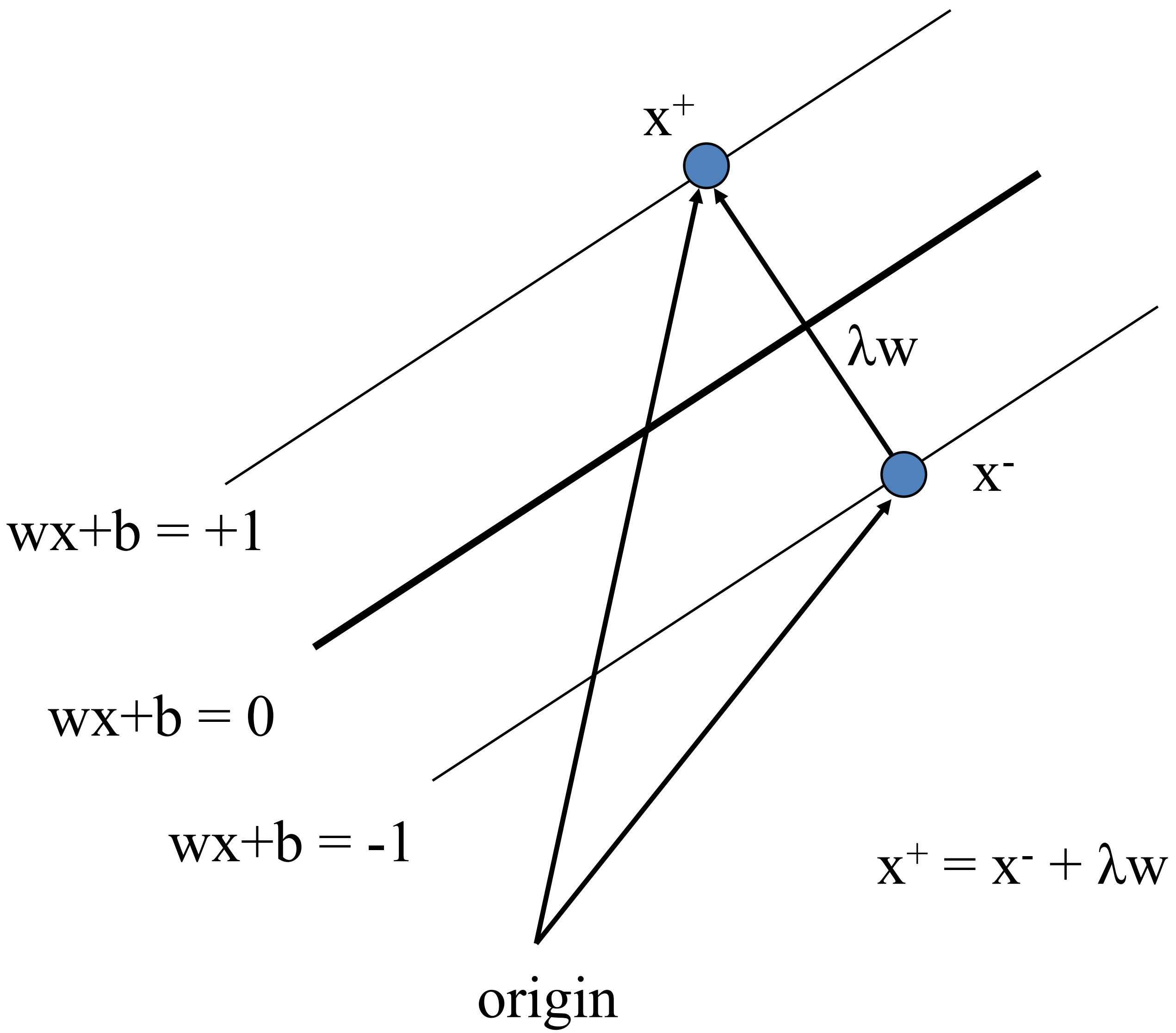
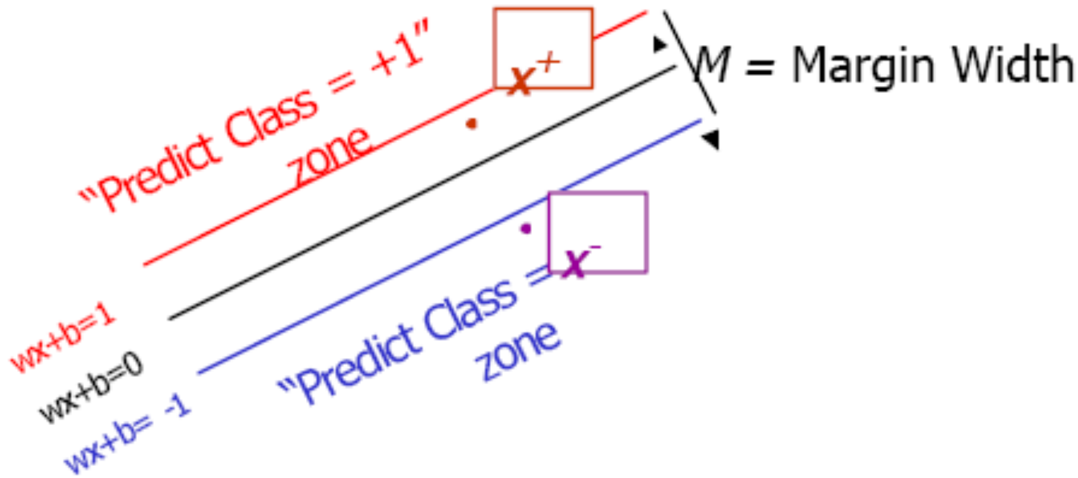
- Minus-plane = $\{ \mathbf{x} : \mathbf{w} \cdot \mathbf{x} + b = -1 \}$
- The vector **w** is perpendicular to the Plus Plane
- Let $\mathbf{x}^-$ be any point on the minus plane
- Let $\mathbf{x}^+$ be the closest plus-plane-point to $\mathbf{x}^-$.
- Claim: $\mathbf{x}^+ = \mathbf{x}^- + \lambda \mathbf{w}$ for some value of λ. Why?

A. Moore, 2003


x^+
λw
x^-
$wx+b = +1$
$wx+b = 0$
$wx+b = -1$
$x^+ = x^- + \lambda w$
origin

Computing the margin width



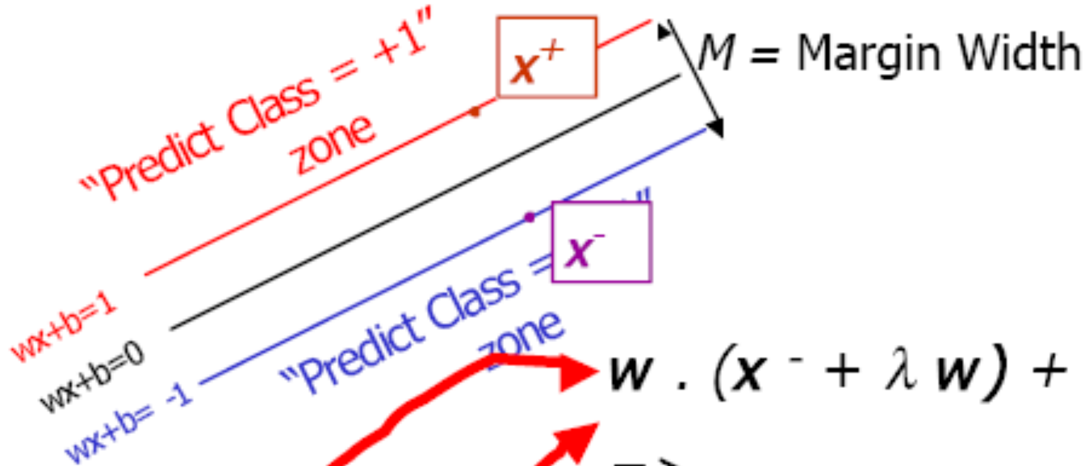
What we know:

- $\mathbf{w} \cdot \mathbf{x}^+ + b = +1$
- $\mathbf{w} \cdot \mathbf{x}^- + b = -1$
- $\mathbf{x}^+ = \mathbf{x}^- + \lambda \mathbf{w}$
- $|\mathbf{x}^+ - \mathbf{x}^-| = M$

It's now easy to get *M* in terms of ***w*** and *b*

A. Moore, 2003

Computing the margin width



What we know:

- $\mathbf{w} \cdot \mathbf{x}^+ + b = +1$
- $\mathbf{w} \cdot \mathbf{x}^- + b = -1$
- $\mathbf{x}^+ = \mathbf{x}^- + \lambda\, \mathbf{w}$
- $|\mathbf{x}^+ - \mathbf{x}^-| = M$

It's now easy to get M in terms of $\mathbf{w}$ and b

$$\mathbf{w} \cdot (\mathbf{x}^- + \lambda\, \mathbf{w}) + b = 1$$

=>

$$\mathbf{w} \cdot \mathbf{x}^- + b + \lambda\, \mathbf{w} \cdot \mathbf{w} = 1$$

=>

$$-1 + \lambda\, \mathbf{w} \cdot \mathbf{w} = 1$$

=>

$$\lambda = \frac{2}{\mathbf{w} \cdot \mathbf{w}}$$

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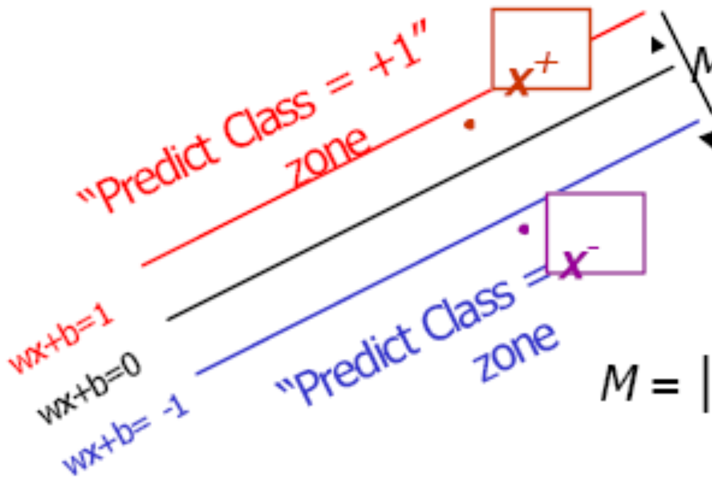
Computing the margin width



M = Margin Width = $\dfrac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$

What we know:

- $\mathbf{w} \,.\, \mathbf{x}^+ + b = +1$
- $\mathbf{w} \,.\, \mathbf{x}^- + b = -1$
- $\mathbf{x}^+ = \mathbf{x}^- + \lambda\, \mathbf{w}$
- $|\mathbf{x}^+ - \mathbf{x}^-| = M$
- $\lambda = \dfrac{2}{\mathbf{w}.\mathbf{w}}$

$$M = |\mathbf{x}^+ - \mathbf{x}^-| = |\lambda\, \mathbf{w}| =$$

$$= \lambda\, |\mathbf{w}| = \lambda \sqrt{\mathbf{w}.\mathbf{w}}$$

$$= \frac{2\sqrt{\mathbf{w}.\mathbf{w}}}{\mathbf{w}.\mathbf{w}} = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$$

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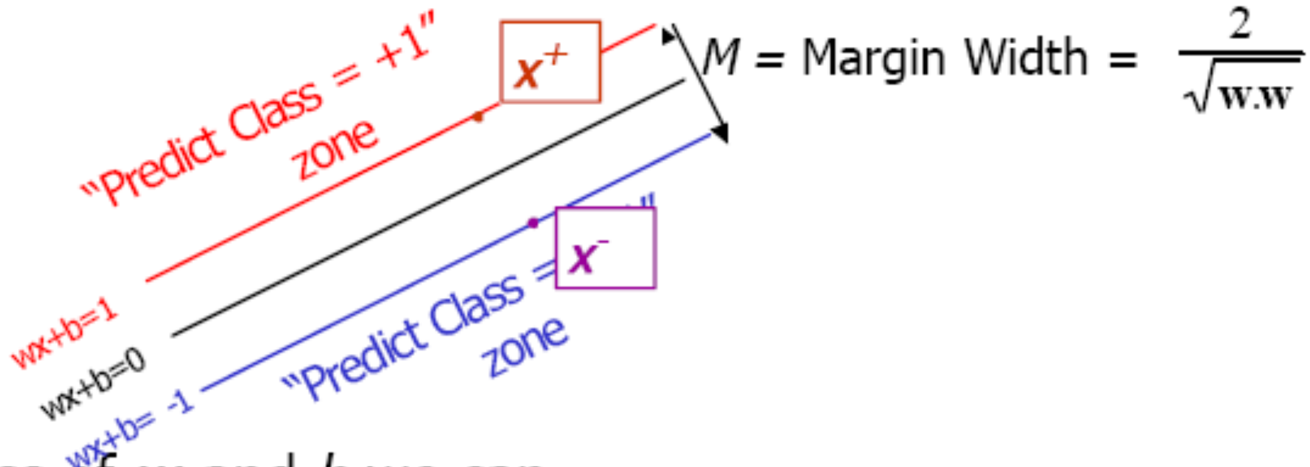
Learning the Maximum Margin Classifier



Given a guess of ***w*** and *b* we can

- Compute whether all data points in the correct half-planes
- Compute the width of the margin

So now we just need to write a program to search the space of **w**'s and *b*'s to find the widest margin that matches all the datapoints. *How?*

Gradient descent? Simulated Annealing? Matrix Inversion? EM? Newton's Method?

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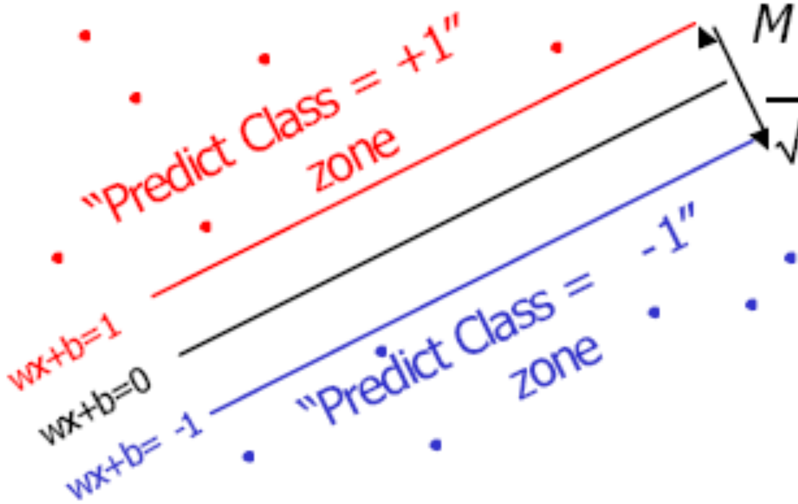
Learning the Maximum Margin Classifier



$M = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$

Given guess of $\mathbf{w}$, b we can

- Compute whether all data points are in the correct half-planes
- Compute the margin width

Assume R datapoints, each $(\mathbf{x}_k, y_k)$ where $y_k = +/- 1$

What should our quadratic optimization criterion be?

How many constraints will we have?

What should they be?

Constrained Optimization Problem

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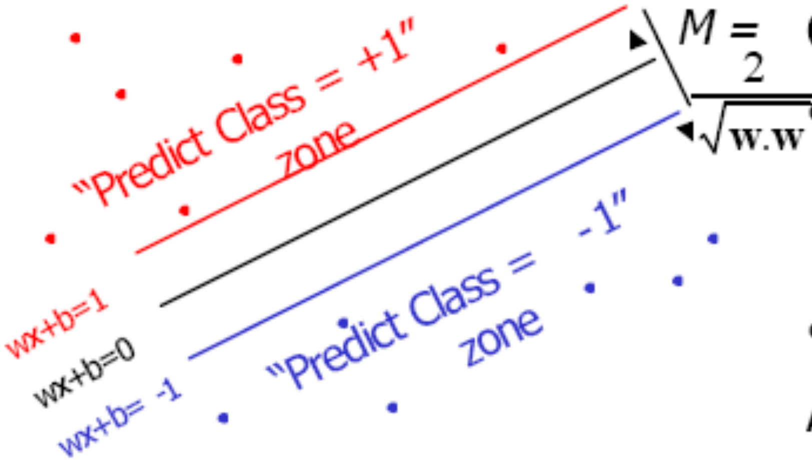
Learning the Maximum Margin Classifier



Given guess of $\mathbf{w}$, b we can

- Compute whether all data points are in the correct half-planes
- Compute the margin width

Assume R datapoints, each $(\mathbf{x}_k, y_k)$ where y_k = +/- 1

What should our quadratic optimization criterion be?

Minimize $\mathbf{w}.\mathbf{w}$

How many constraints will we have? R

What should they be?

$\mathbf{w} \cdot \mathbf{x}_k + b >= 1$ if $y_k = 1$

$\mathbf{w} \cdot \mathbf{x}_k + b <= -1$ if $y_k = -1$

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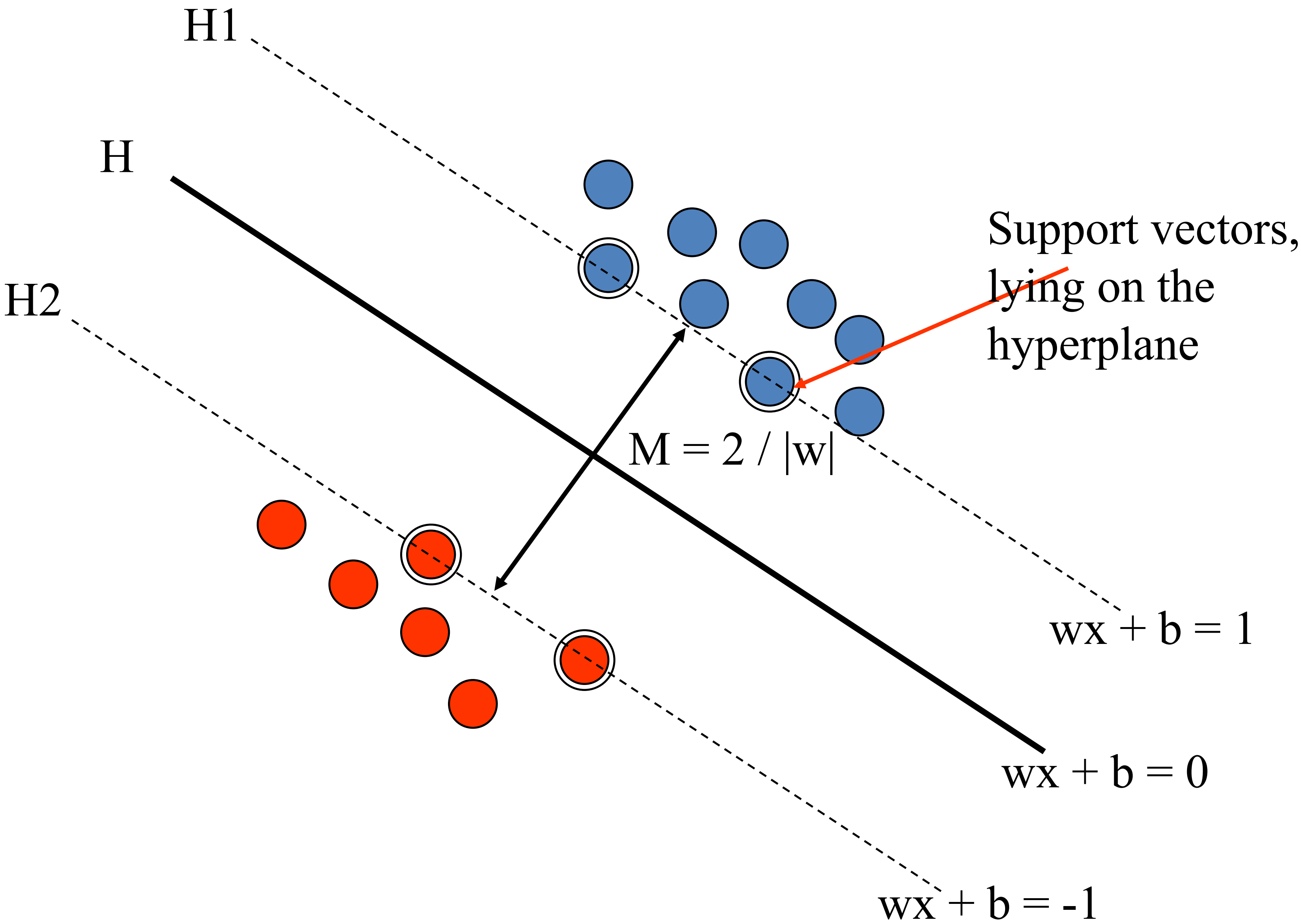
Margin and Support Vectors



Relationship with Vapnik Chevonenkis (VC) Dimension Learning Theory

Expectation of Test Error

$$R(a) = \int \frac{1}{2} | y - f(X, a) | \, p(X, y) dX dy$$

$R(a)$ is called expected risk / loss, the same as before except the ½ ratio.

Empirical Risk $R_{emp}(a)$ is defined to be the measured mean error rate on l training examples.

$$R_{emp}(a) = \frac{1}{2l} \sum_{i=1}^{l} | y_i - f(X_i, a) |$$

Vapnik Risk Bound

- ½$|y_i - f(X_i, a)|$ is also called the loss. It can only take the values 0 and 1.
- Choose η such that 0 <= η <= 1. With probability 1 – η, the following bound holds (Vapnik, 1995)

$$R(a) \leq R_{emp}(a) + \sqrt{\left(\frac{h(\log(2l/h)+1)-\log(\eta/4)}{l}\right)}$$

Where h is a non-negative integer called the Vapnik Chevonenkis (VC) dimension, and is a measure of the notion of capacity. The second part of the right is called VC confidence.

Insights about Risk Bound

- Independent of $p(X,y)$.
- Often not possible to compute the left hand side.
- Easily compute right hand side if h is known.
- **Structural Risk Minimization**: Given sufficiently small η, taking the machine which minimizes the right hand side and gives the lowest upper bound on the actual risk.
- **Question**: how does the bound change according to η?

VC Dimension

- VC dimension is a property of a set of functions { $f(a)$ }. Here we consider functions that correspond to two-class pattern recognition case, so that $f(X,a) \in \{-1, +1\}$.

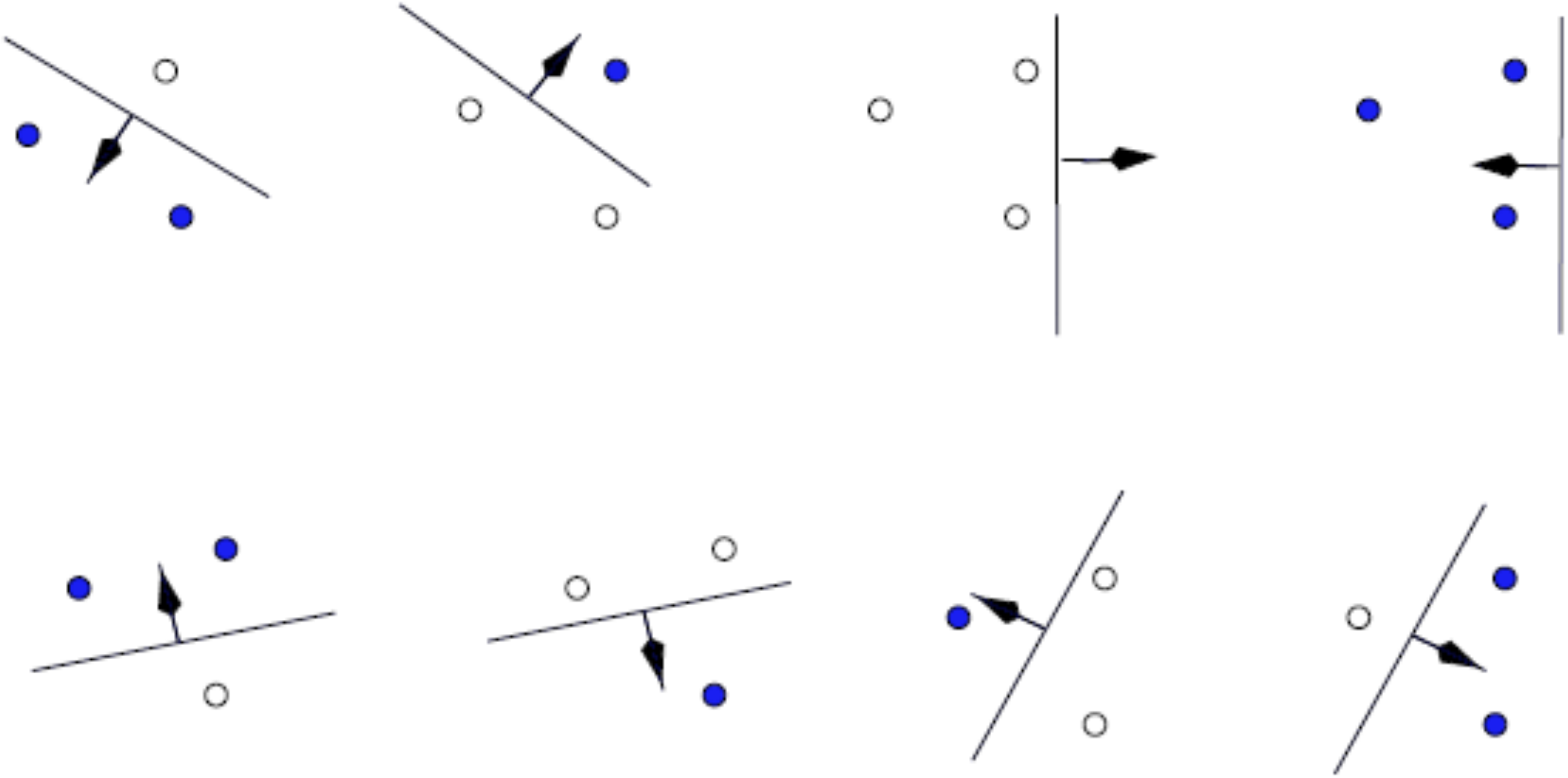
- If a given set of l points can be labeled in all possible 2^l ways, and **for each labeling**, a member of set $\{f(a)\}$ can be found to correctly assign those labels, we say that set of points is shattered by that set of functions.

VC Dimension

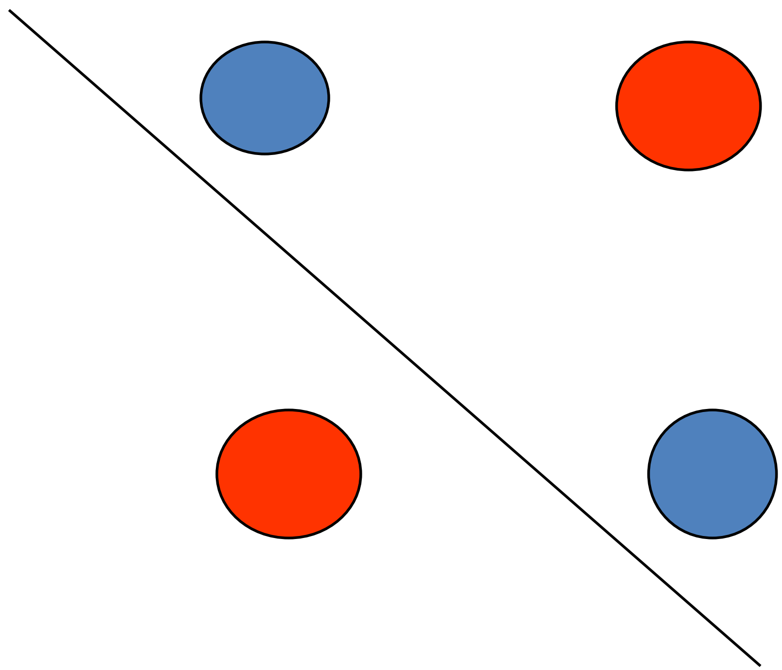
- VC dimension for a set of functions $\{f(a)\}$ is defined as the maximum number of training points that can be shattered by $\{f(a)\}$.

- If the VC dimension is *h*, then there exists at least one set of *h* points that can be shattered. But not necessary for every set of *h* points.

A linear function has VC dimension 3



8 possible labeling of 3 points can be separated by lines.



Simply can not separate the labeling of these four points using a line. So the VC dimension of a line is 3.

VC Dimension and the Number of Parameters

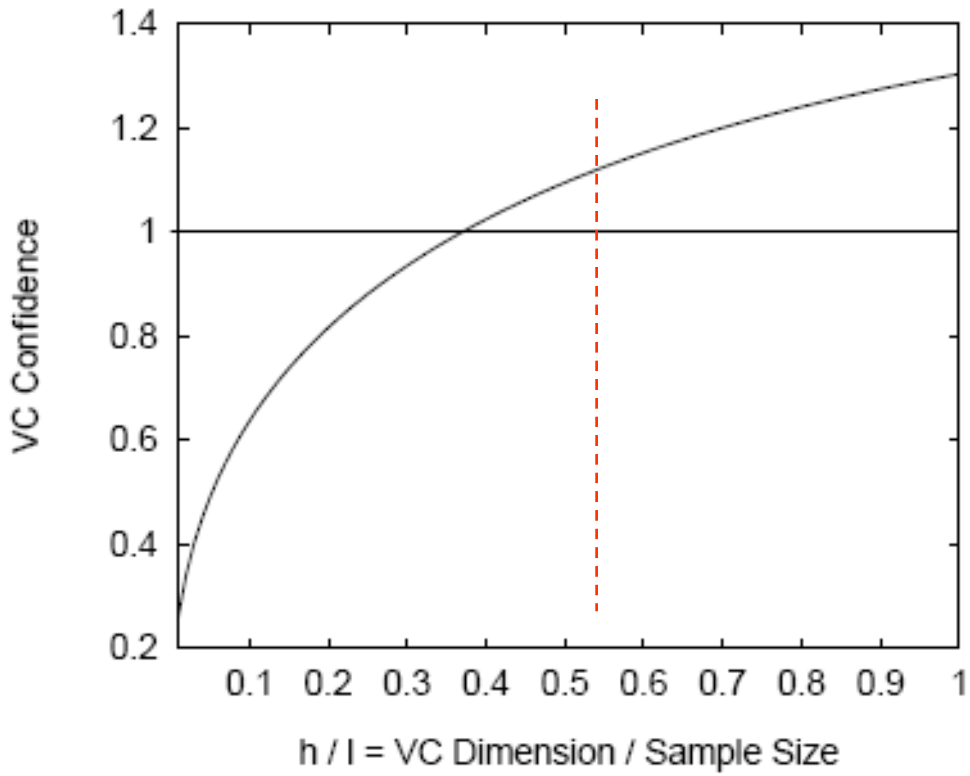
- Intuitively, more parameters ➔ higher VC dimension?
- However, 1 parameter function can have infinite VC dimension. (see Burge's tutorial)

$$f(x, \alpha) \equiv \theta(\sin(\alpha x)), \quad x, \alpha \in \mathbf{R}.$$

If $\sin(ax) > 0$, $f(x,a) = 1$, -1 otherwise

VC Confidence and VC Dimension *h*

$$R(a) \leq R_{emp}(a) + \sqrt{\left(\frac{h(\log(2l/h)+1) - \log(\eta/4)}{l}\right)}$$

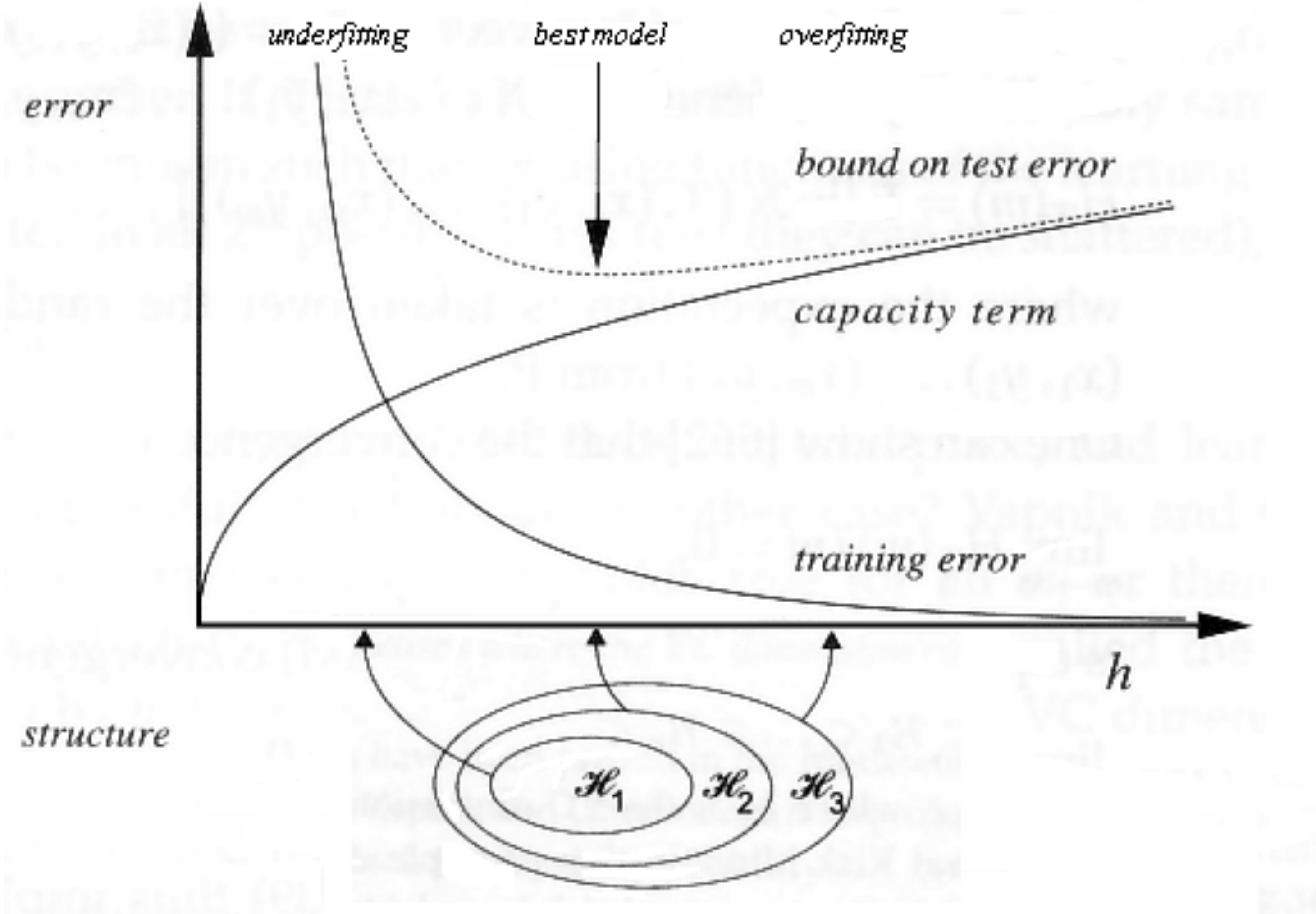


VC confidence is monotonic in h. (here $l = 10{,}000$, $\eta = 0.05$ (95%))

Structural Risk Minimization

$$R(a) \leq R_{emp}(a) + \sqrt{\left(\frac{h(\log(2l/h)+1) - \log(\eta/4)}{l}\right)}$$

Given some selection of learning machines whose empirical risk is zero, one wants to choose that learning machine whose associated set of functions has minimal VC dimension. This is called **Occam's Razor**: "**All things being equal, the simplest solution tends to be the best one**."



http://www.svms.org/srm/

Comments

- The risk bound equation gives a probabilistic upper bound on the actual risk. This does not prevent a particular machine with the same value for empirical risk, and whose function set has higher VC dimension from having better performance.
- For higher h value, the bound is guaranteed not tight.
- $h/l > 0.37$, VC confidence exceeds unity.

Example

- What is the VC dimension of one-nearest neighbor method?
- Nearest neighbor classifier can still perform well.
- For any classifier with an infinite VC dimension, the bound is not even valid.

Structure Risk Minimization for SVM

- Margin (M) is a measure of capacity / complexity of a linear support vector machine
- The objective is to find a linear hyperplane with maximum margin
- Maximum margin classifier

Maximum Margin Classifier

- The optimization problem:

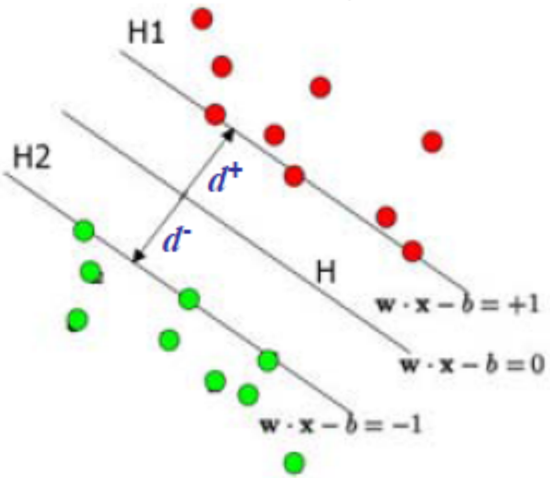
$$\max_{w,b} \quad \frac{1}{\|w\|}$$

$$\text{s.t} \quad y_i(w^T x_i + b) \geq 1, \quad \forall i$$

- The solution to this leads to the famous Support Vector Machines --- believed by many to be the best "off-the-shelf" supervised learning algorithm

Support Vector Machines Optimization

- A convex quadratic programming problem with linear constrains:



$$\max_{w,b} \quad \frac{1}{\|w\|}$$

$$\text{s.t} \quad y_i(w^T x_i + b) \geq 1, \quad \forall i$$

  - The attained margin is now given by $\frac{1}{\|w\|}$
  - Only a few of the classification constraints are relevant ➔ **support vectors**

- Constrained optimization
  - We can directly solve this using commercial quadratic programming (QP) code
  - But we want to take a more careful investigation of Lagrange duality, and the solution of the above in its dual form.

➔ deeper insight: support vectors, kernels …

➔ more efficient algorithm

Lagrange Optimization

- An mathematical optimization technique named after **Joseph Louis Lagrange**
- A method for finding local minima of a function of several variables subject to one or more constraints
- The method reduces a problem in ***n*** variables with ***k*** constraints to a *solvable* problem in ***n+k*** variables with no constraints.
- The method introduces a new unknown scalar variable, the **Lagrange multiplier**, for each constraint and forms a linear combination involving the multipliers as coefficients.

http://en.wikipedia.org/wiki/Lagrange_multipliers

Langrangian Duality

- The Primal Problem

Primal:

$$\begin{aligned} \min_w \quad & f(w) \\ \text{s.t.} \quad & g_i(w) \le 0, \quad i = 1, \ldots, k \\ & h_i(w) = 0, \quad i = 1, \ldots, l \end{aligned}$$

The generalized Lagrangian:

$$\mathcal{L}(w, \alpha, \beta) = f(w) + \sum_{i=1}^{k} \alpha_i g_i(w) + \sum_{i=1}^{l} \beta_i h_i(w)$$

the α's ($\alpha_i \geq 0$) and β's are called the Lagarangian multipliers

Lemma:

$$\max_{\alpha, \beta, \alpha_i \ge 0} \mathcal{L}(w, \alpha, \beta) = \begin{cases} f(w) & \text{if } w \text{ satisfies primal constraints} \\ \infty & \text{o/w} \end{cases}$$

A re-written Primal:

$$\min_w \max_{\alpha, \beta, \alpha_i \ge 0} \mathcal{L}(w, \alpha, \beta)$$

Lagrangian Duality

- Recall the Primal Problem:

$$\min_w \max_{\alpha,\beta,\alpha_i \geq 0} \mathcal{L}(w,\alpha,\beta)$$

- The Dual Problem:

$$\max_{\alpha,\beta,\alpha_i \geq 0} \min_w \mathcal{L}(w,\alpha,\beta)$$

- **Theorem (weak duality):**

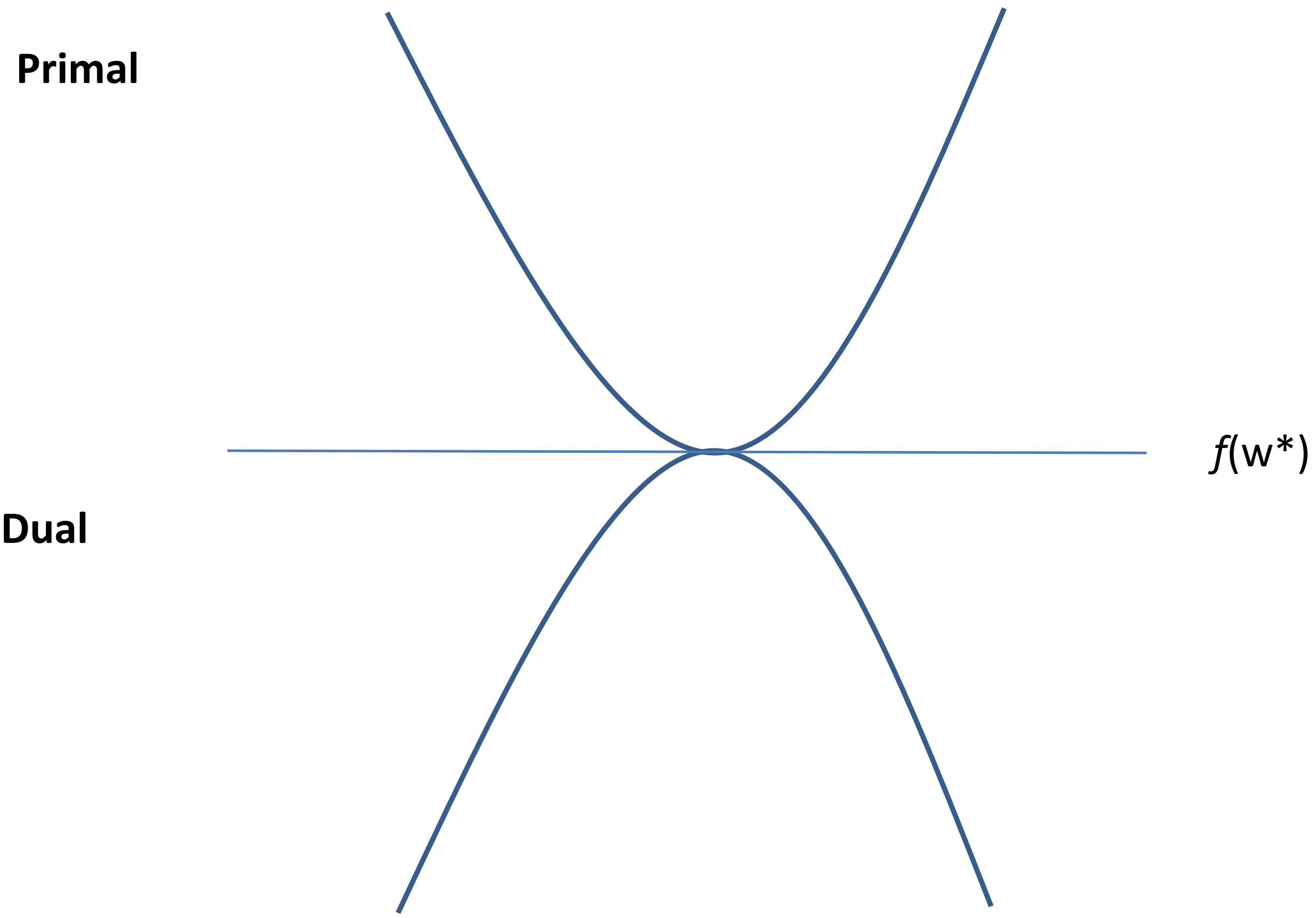
$$d^* = \max_{\alpha,\beta,\alpha_i \geq 0} \min_w \mathcal{L}(w,\alpha,\beta) \leq \min_w \max_{\alpha,\beta,\alpha_i \geq 0} \mathcal{L}(w,\alpha,\beta) = p^*$$

- **Theorem (strong duality):**

  Iff there exist a saddle point of $\mathcal{L}(w,\alpha,\beta)$, we have

$$d^* = p^*$$

Primal and Dual Problems



KKT Condition

- If there exists some saddle point of $\mathcal{L}$, then the saddle point satisfies the following "Karush-Kuhn-Tucker" (KKT) conditions:

$$\frac{\partial}{\partial w_i}\mathcal{L}(w,\alpha,\beta)=0,\quad i=1,\dots,k$$

$$\frac{\partial}{\partial \beta_i}\mathcal{L}(w,\alpha,\beta)=0,\quad i=1,\dots,l$$

$$\alpha_i g_i(w)=0,\quad i=1,\dots,m$$ Complementary slackness

$$g_i(w)\le 0,\quad i=1,\dots,m$$ Primal feasibility

$$\alpha_i \ge 0,\quad i=1,\dots,m$$ Dual feasibility

- **Theorem**: If w^*, α^* and β^* satisfy the KKT condition, then it is also a solution to the primal and the dual problems.

Solve Maximum Margin Classifier

- Recall our opt problem:

$$\begin{aligned} &\max_{w,b} \quad \frac{1}{\|w\|} \\ &\text{s.t} \quad y_i(w^T x_i + b) \geq 1, \quad \forall i \end{aligned}$$

- This is equivalent to

$$\begin{aligned} &\min_{w,b} \quad \frac{1}{2} w^T w \\ &\text{s.t} \quad 1 - y_i(w^T x_i + b) \leq 0, \quad \forall i \end{aligned} \qquad (*)$$

- Write the Lagrangian:

$$\mathcal{L}(w,b,\alpha) = \frac{1}{2} w^T w - \sum_{i=1}^{m} \alpha_i \left[y_i (w^T x_i + b) - 1 \right]$$

  - Recall that (*) can be reformulated as $\min_{w,b} \max_{\alpha_i \geq 0} \mathcal{L}(w,b,\alpha)$
    Now we solve its **dual problem**: $\max_{\alpha_i \geq 0} \min_{w,b} \mathcal{L}(w,b,\alpha)$

The Dual Problem

$$\max_{\alpha_i \geq 0} \min_{w,b} \mathcal{L}(w,b,\alpha)$$

- We minimize $\mathcal{L}$ with respect to w and b first:

$$\nabla_w \mathcal{L}(w,b,\alpha) = w - \sum_{i=1}^{m} \alpha_i y_i x_i = 0, \qquad (*)$$

$$\nabla_b \mathcal{L}(w,b,\alpha) = \sum_{i=1}^{m} \alpha_i y_i = 0, \qquad (**)$$

Note that (*) implies:

$$w = \sum_{i=1}^{m} \alpha_i y_i x_i \qquad (***)$$

- Plus (***) back to $\mathcal{L}$, and using (**), we have:

$$\mathcal{L}(w,b,\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j (\mathbf{x}_i^T \mathbf{x}_j)$$

Proof

$$\frac{1}{2}w^T w - \sum_{i=1}^{m} a_i(y_i(w^T x_i + b) \; - 1)$$

Replace w with $\sum_{i=1}^{m} a_i y_i x_i$, we get:

$$=\frac{1}{2}\sum_{i=1}^{m} a_i y_i x_i \sum_{i=1}^{m} a_i y_i x_i - \sum_{i=1}^{m} a_i(y_i((\sum_{i=1}^{m} a_i y_i x_i)x_i + b) - 1)$$

$$= \frac{1}{2}\sum_{i=1}^{m} a_i y_i x_i \sum_{i=1}^{m} a_i y_i x_i - (\sum_{i=1}^{m} a_i y_i(\sum_{i=1}^{m} a_i y_i x_i)x_i + \sum_{i=1}^{m} a_i y_i \, b - \sum_{i=1}^{m} a_i)$$

$$= -\frac{1}{2}\sum_{i,j=1}^{m} a_i a_j y_i y_j x_i^T x_j \quad + \sum_{i=1}^{m} a_i$$

The Dual Problem

- Now we have the following dual opt problem:

$$\max_\alpha \mathcal{J}(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j (\mathbf{x}_i^T \mathbf{x}_j)$$

$$\text{s.t.} \quad \alpha_i \geq 0, \quad i = 1, \ldots, k$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0.$$

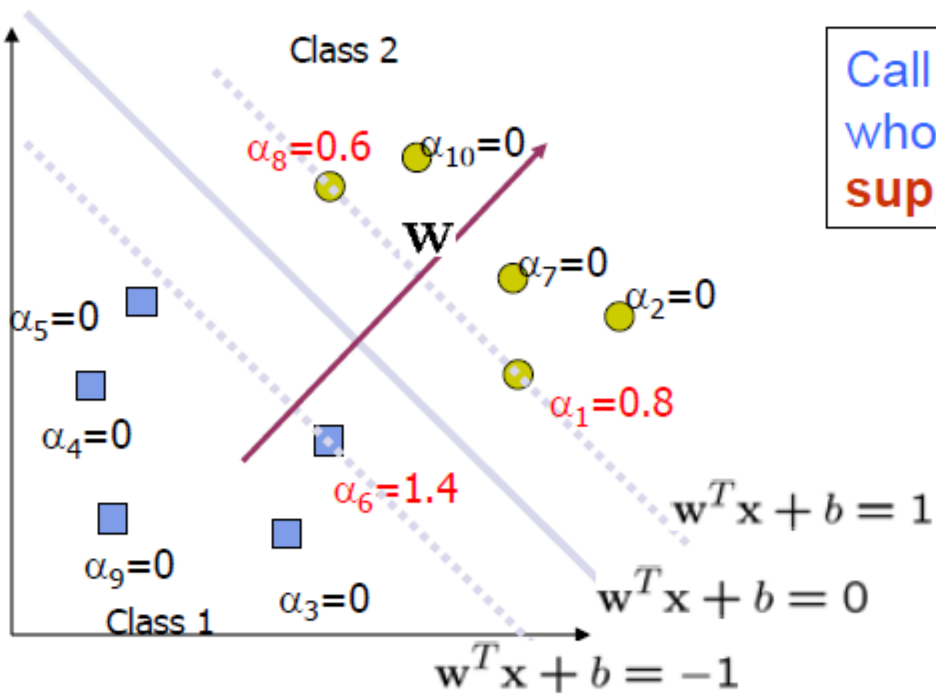
- This is, (again,) a **quadratic programming** problem.
  - A global maximum of α_i can always be found.
  - But what's the big deal??
  - Note two things:

1. w can be recovered by $\quad w = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i \quad$ See next …
2. The "kernel" $\quad \mathbf{x}_i^T \mathbf{x}_j \quad$ More later …

Support Vectors

- Note the KKT condition --- only a few α_i's can be nonzero!!

$$\alpha_i g_i(w) = 0, \quad i = 1, \ldots, m$$



Call the training data points whose α_i's are nonzero the **support vectors** (SV)

Support Vector Machines

- Once we have the Lagrange multipliers $\{\alpha_i\}$, we can reconstruct the parameter vector w as a weighted combination of the training examples:

$$w = \sum_{i \in SV} \alpha_i y_i \mathbf{x}_i$$

Question: how to get b?

- For testing with a new data z
  - Compute
  $$w^T z + b = \sum_{i \in SV} \alpha_i y_i \left(\mathbf{x}_i^T z \right) + b$$
  and classify z as class 1 if the sum is positive, and class 2 otherwise
  - Note: w need not be formed explicitly

How to Determine w and b

- Use quadratic programming to solve a_i and compute w is trivial. (use KKT condition (1))
- How to compute b?
- Use KKT condition (5), for any support vector (point $a_i > 0$), $y_i(w.x_i+b)-1 = 0$.
- We compute b in terms of a support vector. Better: we computer b in terms of all support vectors and take the average.

Interpretation of Support Vector Machines

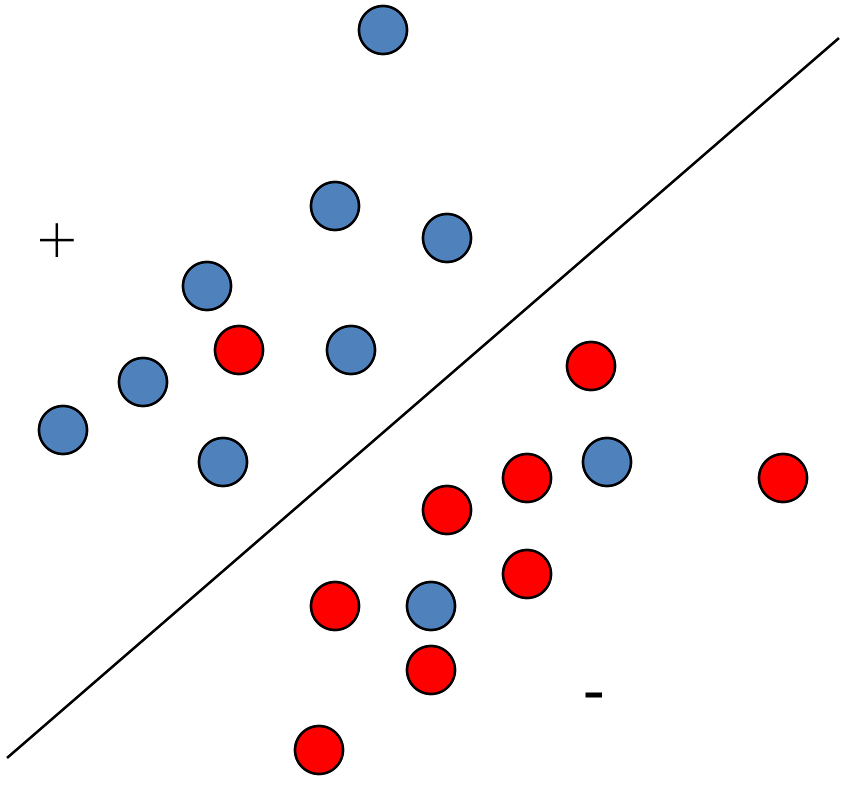
- The optimal $\boldsymbol{w}$ is a linear combination of a small number of data points. This "sparse" representation can be viewed as data compression as in the construction of kNN classifier

- To compute the weights $\{\alpha_i\}$, and to use support vector machines we need to specify only the inner products (or kernel) between the examples $\mathbf{x}_i^T \mathbf{x}_j$

- We make decisions by comparing each new example z with only the support vectors:

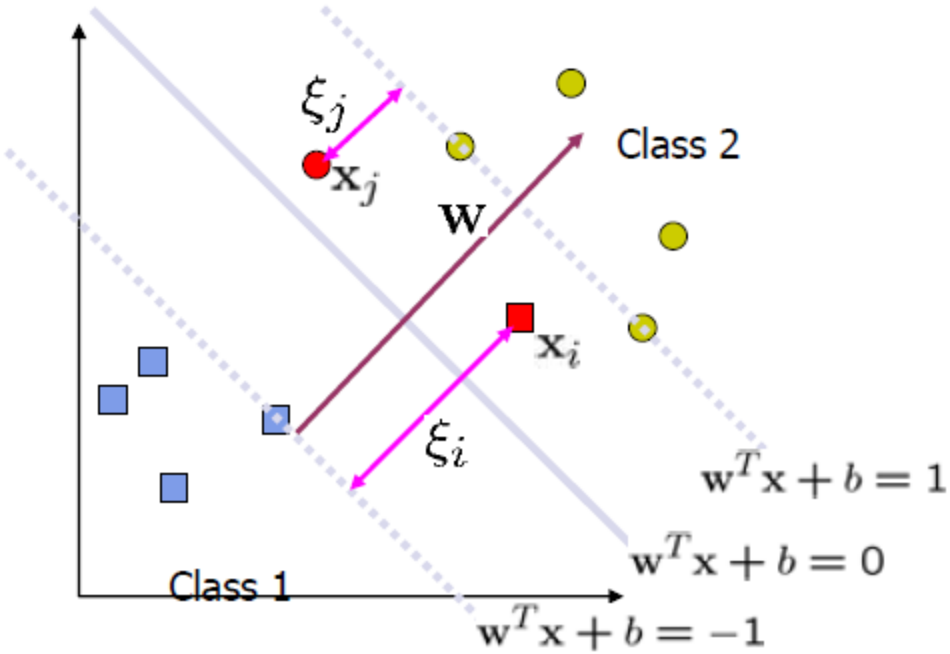
$$y^* = \operatorname{sign}\left(\sum_{i \in SV} \alpha_i y_i \left(\mathbf{x}_i^T z \right) + b \right)$$

Non-Separable Case



Can't satisfy the constraints $y_i(wx_i+b) >= 1$ for some data points? What can we do?

Non-Linearly Separable Problem



- We allow "error" ξ_i in classification; it is based on the output of the discriminant function $\boldsymbol{w}^T\boldsymbol{x}+b$
- ξ_i approximates the number of misclassified samples

Relax Constraints – Soft Margin

- Introduce positive slack variables ξ_i, i = 1, ..., l to relax constraints. ($\xi_i >= 0$)
- New constraints:
- $x_i.w + b >= +1 - \xi_i$ for $y_i = +1$
- $x_i.w + b <= -1 + \xi_i$ for $y_i = -1$
- Or $\mathbf{y_i(wx_i + b) >= 1 - \xi_i}$
- $\xi_i >= 0$
- For an classification error to happen, the corresponding ξ_i must exceed unity, so **$\Sigma\ \xi_i$ is an upper bound on the number of training errors**.

Soft Margin Hyperplane

- Now we have a slightly different opt problem:

$$\min_{w,b} \quad \frac{1}{2} w^T w + C \sum_{i=1}^{m} \xi_i$$

$$\text{s.t} \quad \begin{aligned} & y_i(w^T x_i + b) \geq 1 - \xi_i, \quad \forall i \\ & \xi_i \geq 0, \quad \forall i \end{aligned}$$

- ξ_i are "slack variables" in optimization
- Note that ξ_i=0 if there is no error for $\mathbf{x}_i$
- ξ_i is an upper bound of the number of errors
- C : tradeoff parameter between error and margin

Primal Optimization

$$L_P = \frac{1}{2}|w|^2 + C\sum_i \xi_i - \sum_{i=1}^{m} a_i(y_i(x_i.w+b)-1+\xi_i)) - \sum_{i=1}^{m} u_i\xi_i$$

$$\frac{\partial L_p}{\partial \xi_i} = C - a_i - u_i = 0$$

$$\Rightarrow a_i <= C$$

u_i is the Lagrange multipliers introduced to enforce Non-negativity of ξ_i

KKT Conditions

$$
\begin{array}{ll}
1.\partial_{\mathbf{w}}\mathcal{L}_P = 0 \quad \rightarrow & \mathbf{w} - \sum_i \alpha_i y_i \mathbf{x}_i = 0 \\
2.\partial_b\mathcal{L}_P = 0 \quad \rightarrow & \sum_i \alpha_i y_i = 0 \\
3.\partial_\xi\mathcal{L}_P = 0 \quad \rightarrow & C - \alpha_i - \mu_i = 0 \\
4.\text{constraint-1} & y_i(\mathbf{w}^T\mathbf{x}_i - b) - 1 + \xi_i \geq 0 \\
5.\text{constraint-2} & \xi_i \geq 0 \\
6.\text{multiplier condition-1} & \alpha_i \geq 0 \\
7.\text{multiplier condition-2} & \mu_i \geq 0 \\
8.\text{complementary slackness-1} & \alpha_i \left[y_i(\mathbf{w}^T\mathbf{x}_i - b) - 1 + \xi_i\right] = 0 \\
9.\text{complementary slackness-1} & \mu_i \xi_i = 0
\end{array}
$$

Max Welling, 2005

Proof of Soft Margin Optimization

$$L_P = \frac{1}{2}|w|^2 + C\sum_i \xi_i - \sum_{i=1}^{m} a_i(y_i(x_i.w+b)-1+\xi_i)) - \sum_{i=1}^{m} u_i\xi_i$$

$$= \frac{1}{2}|w|^2 - \sum_{i=1}^{m} a_i(y_i(x_i.w+b)-1) - \sum_{i=1}^{m}(C-a_i-u_i)\xi_i$$

$$= ?$$

The Optimization Problem

- The dual of this new constrained optimization problem is

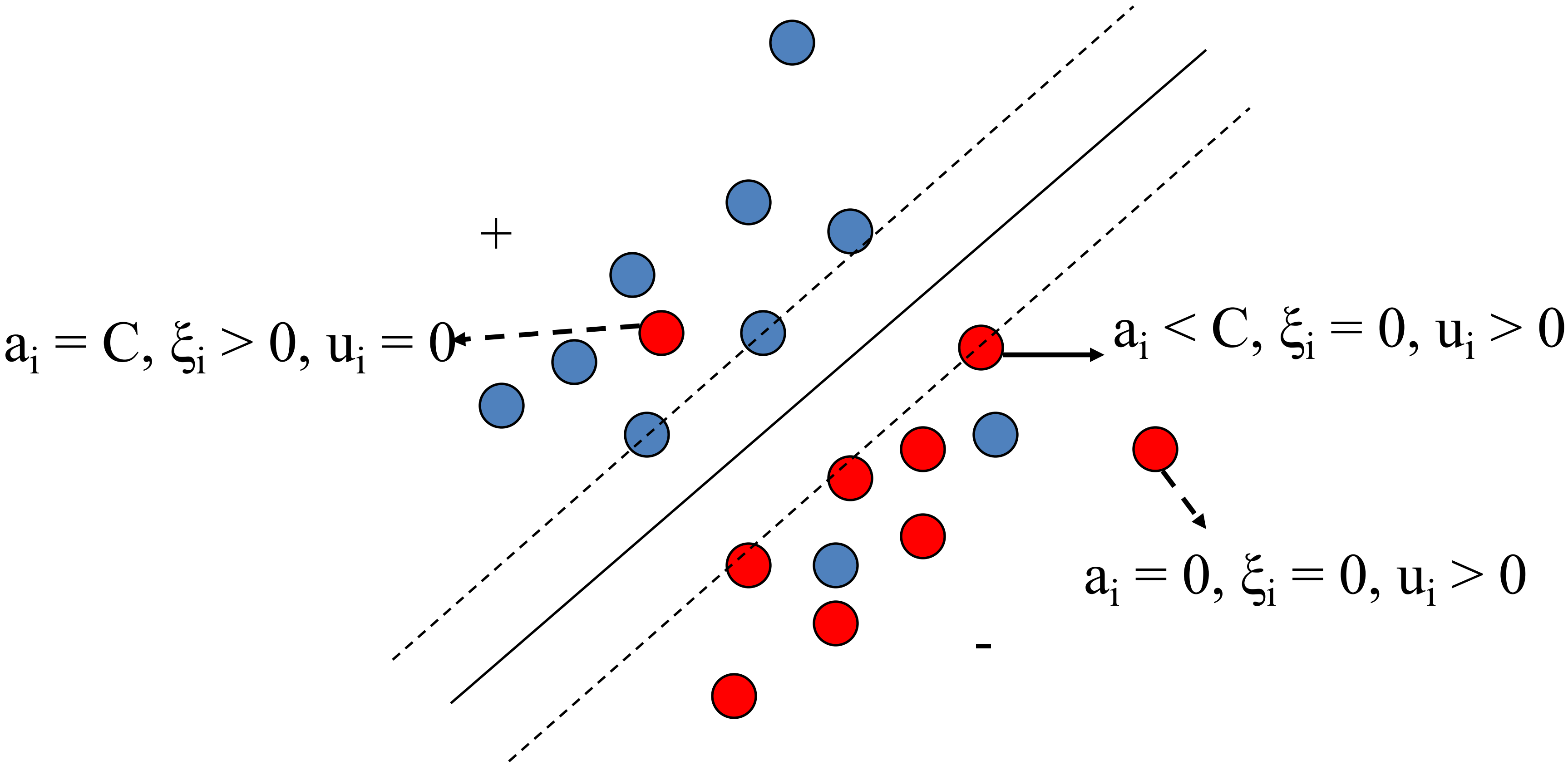
$$\max_{\alpha} \quad \mathcal{J}(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j (\mathbf{x}_i^T \mathbf{x}_j)$$

$$\text{s.t.} \quad 0 \leq \alpha_i \leq C, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0.$$

- This is very similar to the optimization problem in the linear separable case, except that there is an upper bound *C* on α_i now
- Once again, a QP solver can be used to find α_i

Values of Multipliers



Solution of w and b

$$w = \sum_{i=1}^{Ns} a_i y_i x_i$$

Use complementary slackness to compute b. Choose a support vector ($0 < a_i < C$) to compute b, where $\xi_i = 0$. $\xi_i = 0$ is derived by combining equations 3 and 9.

New Objective Function

- Minimize $|w|^2/2 + C(\Sigma\xi_i)^k$.
- C is parameter to be chosen by the user, a larger C corresponding to assigning a higher penalty to errors.
- This is a convex programming problem for any positive integer k.

SVM Demo

https://www.youtube.com/watch?v=bqwAlpumoPM

http://cs.stanford.edu/people/karpathy/svmj
s/demo/