# Model Selection

Dr. Jianlin Cheng

Department of Electrical Engineering
and Computer Science Department
University of Missouri, Columbia

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Slides Adapted from Book and CMU, Stanford Machine Learning Courses

# True vs. Empirical Risk

True Risk: Target performance measure

Classification – Probability of misclassification $P(f(X) \neq Y)$

Regression – Mean Squared Error $\mathbb{E}[(f(X) - Y)^2]$

performance on a random test point (X,Y)

Empirical Risk: Performance on training data

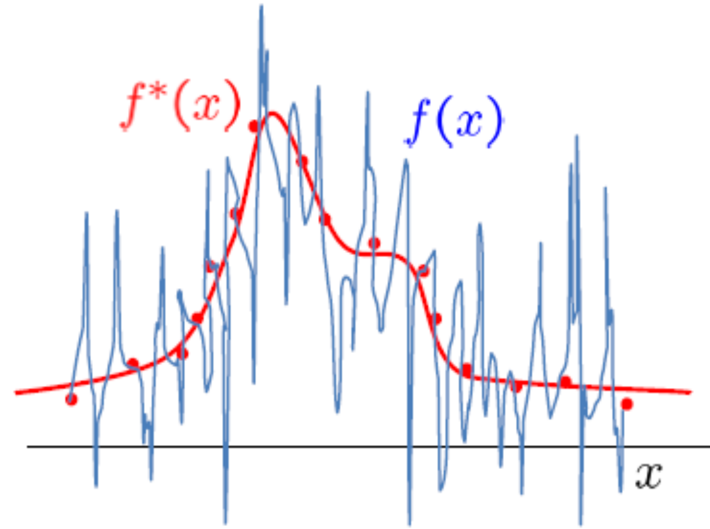
Classification – Proportion of misclassified examples $\frac{1}{n}\sum_{i=1}^{n} 1_{f(X_i) \neq Y_i}$

Regression – Average Squared Error $\frac{1}{n}\sum_{i=1}^{n} (f(X_i) - Y_i)^2$

# Overfitting

Is the following predictor a good one?

$$f(x) = \begin{cases} Y_i, & x = X_i \text{ for } i = 1, \ldots, n \\ \text{any value}, & \text{otherwise} \end{cases}$$




What is its empirical risk? (performance on training data)

zero !

What about true risk?

> zero

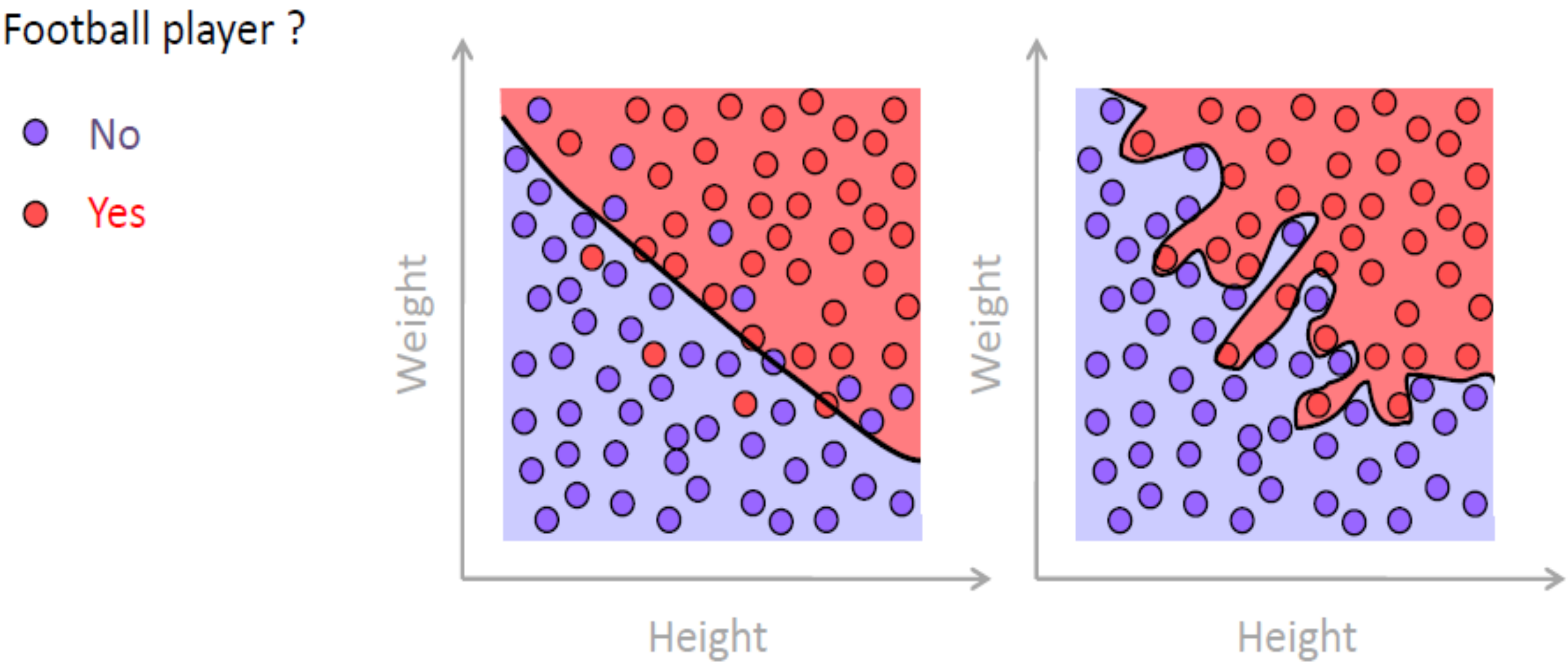
Will predict very poorly on new random test point:

Large generalization error !

# Overfitting

If we allow very complicated predictors, we could overfit the training data.

Examples: Classification 1-NN classifier

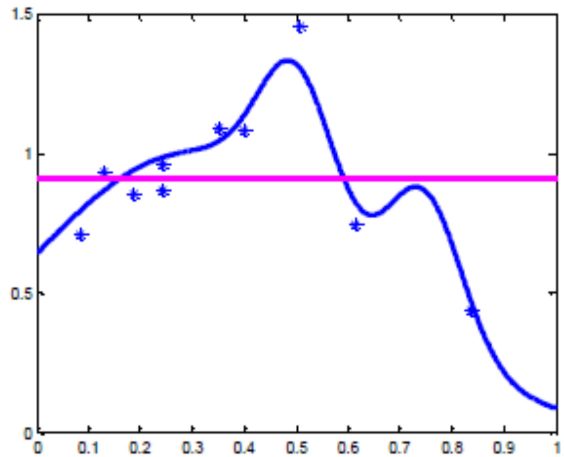


# Overfitting

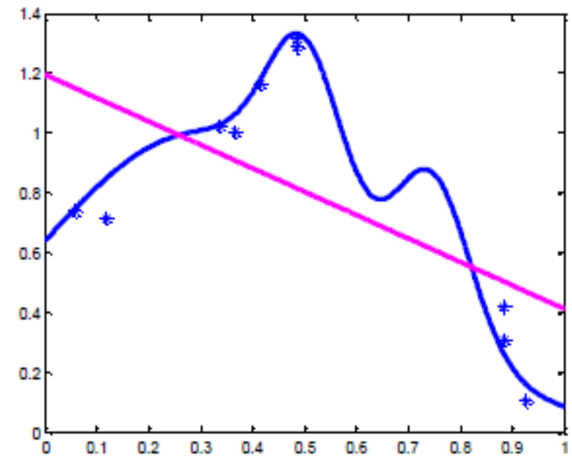
If we allow very complicated predictors, we could overfit the training data.

Examples: Regression (Polynomial of order k – degree up to k-1)

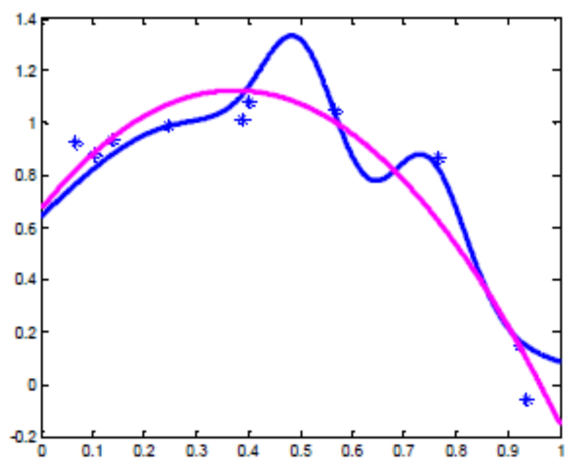
k=1



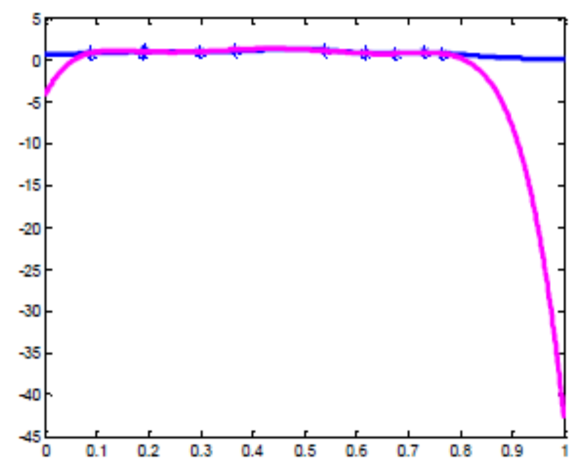
k=2



k=3

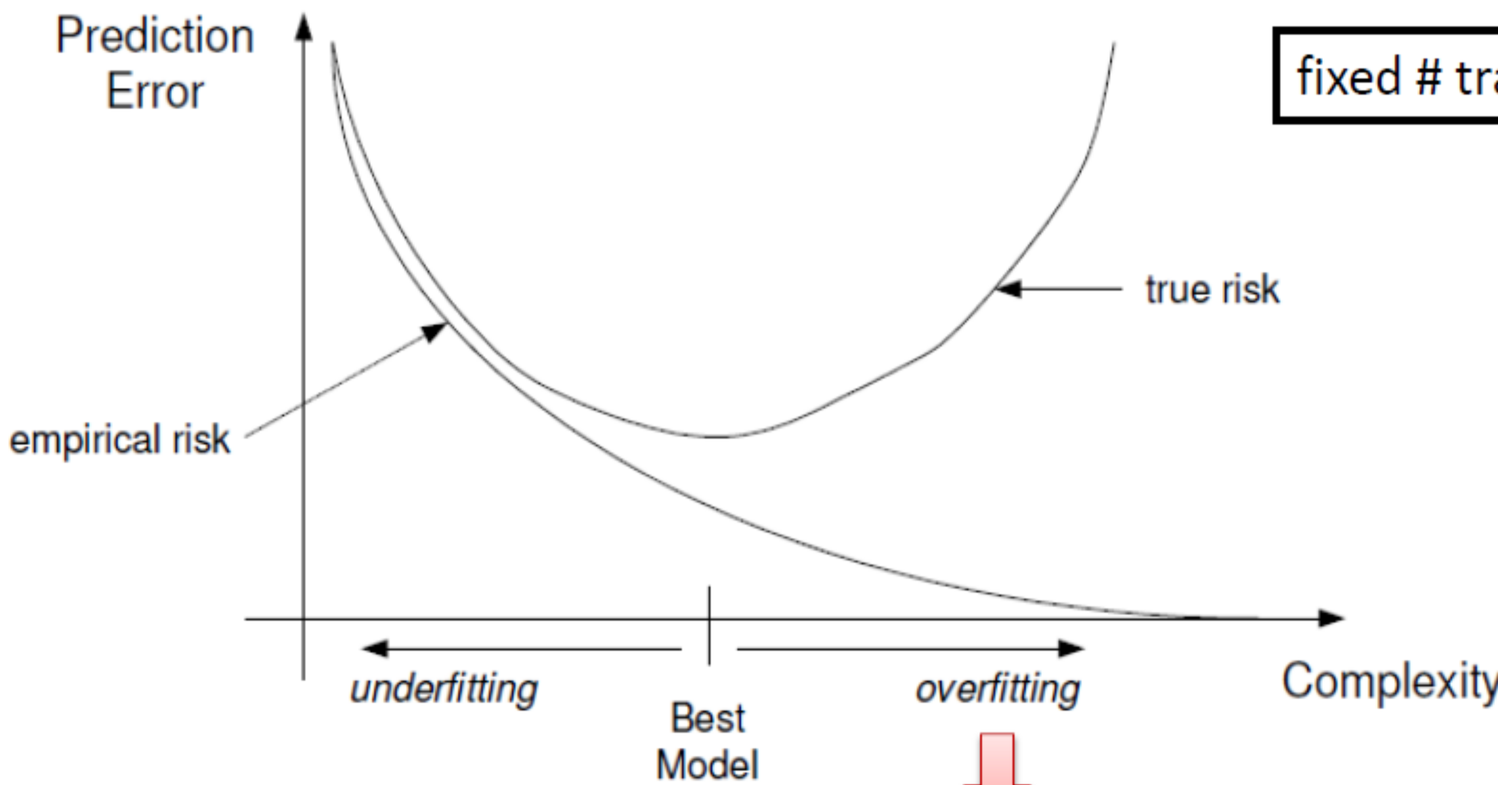


k=7



# Effect of Model Complexity

If we allow very complicated predictors, we could overfit the training data.




fixed # training data

Empirical risk is no longer a good indicator of true risk

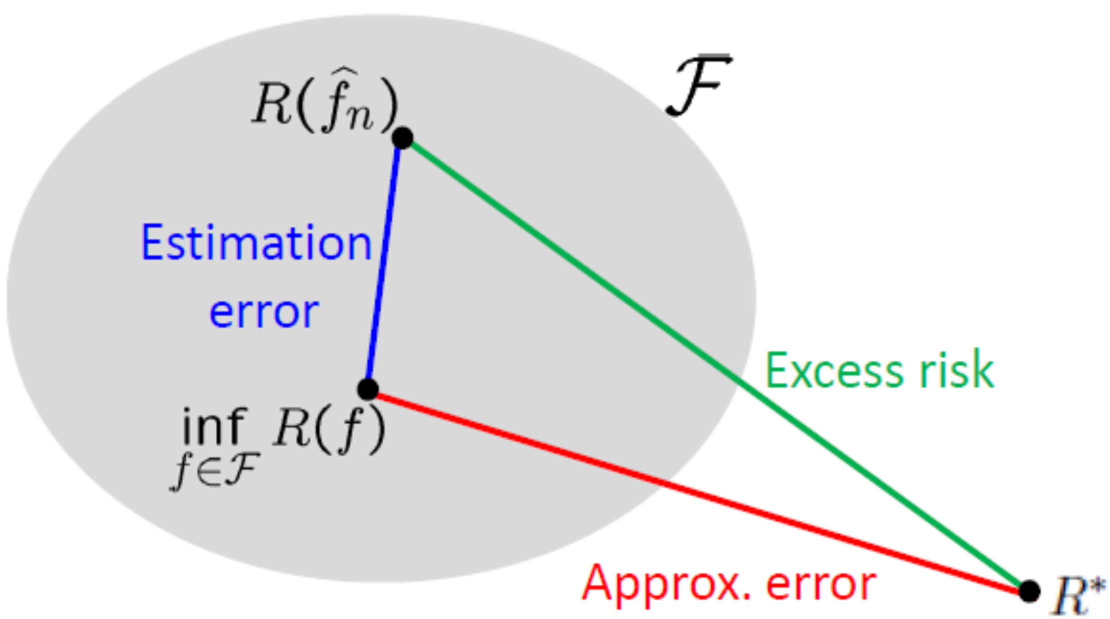
# Behavior of True Risk

Want $\widehat{f}_n$ to be as good as optimal predictor $f^*$

Excess Risk $E\left[R(\widehat{f}_n)\right] - R^* = \underbrace{\left(E[R(\widehat{f}_n)] - \inf_{f\in\mathcal{F}} R(f)\right)}_{\text{estimation error}} + \underbrace{\left(\inf_{f\in\mathcal{F}} R(f) - R^*\right)}_{\text{approximation error}}$

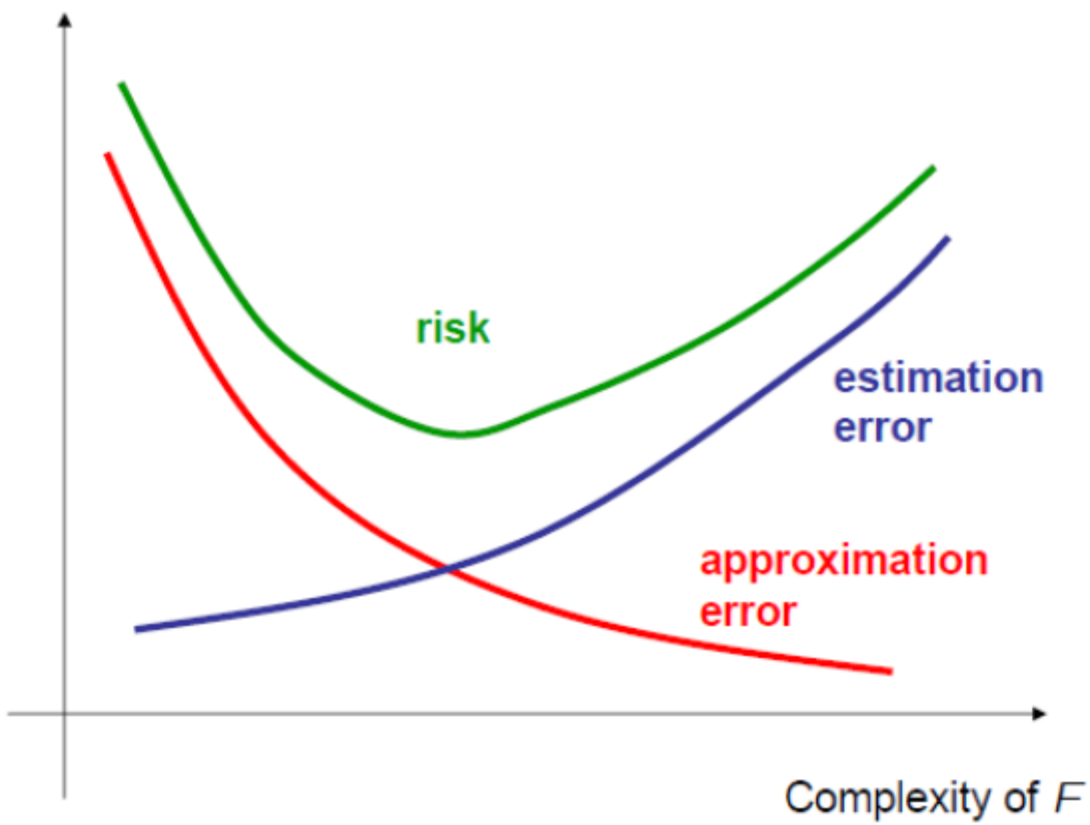
**finite sample size + noise** ← Due to randomness of training data

Due to restriction of model class



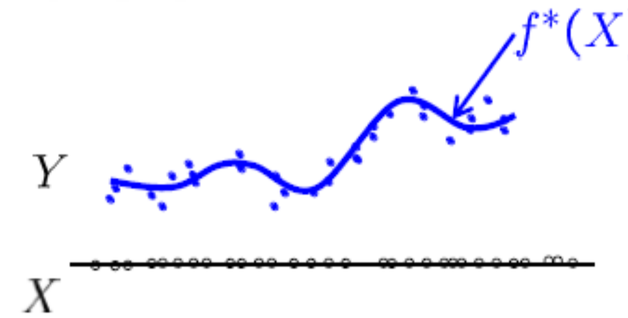
# Behavior of True Risk

$$E\left[R(\widehat{f}_n)\right] - R^* \;=\; \underbrace{\left(E[R(\widehat{f}_n)] - \inf_{f\in\mathcal{F}} R(f)\right)}_{\text{estimation error}} + \underbrace{\left(\inf_{f\in\mathcal{F}} R(f) - R^*\right)}_{\text{approximation error}}$$



# Bias – Variance Tradeoff

Regression: $Y = f^*(X) + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \sigma^2)$




$$R^* = \mathbb{E}_{XY}[(f^*(X) - Y)^2] = \mathbb{E}[\epsilon^2] = \sigma^2$$

Notice: Optimal predictor does not have zero error

$$\mathbb{E}_{D_n}[R(\widehat{f}_n)] = \mathbb{E}_{X,Y,D_n}[(\widehat{f}_n(X) - Y)^2]$$

$D_n$ - training data of size $n$

$$\vdots$$

$$= \underbrace{\mathbb{E}_{X,Y,D_n}[(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)])^2]}_{\text{variance}} + \underbrace{\mathbb{E}_{X,Y}[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))^2]}_{\text{bias}^\wedge 2} + \underbrace{\sigma^2}_{\text{Noise var}}$$

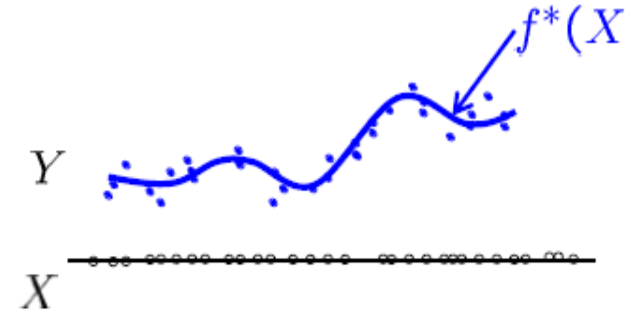
Excess Risk = $\mathbb{E}_{D_n}[R(\widehat{f}_n)] - R^*$ = variance + bias^2

Random component $\equiv$ est err (variance)

$\equiv$ approx err (bias^2)

# Bias – Variance Tradeoff: Derivation

Regression: $Y = f^*(X) + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \sigma^2)$




$$R^* = \mathbb{E}_{XY}[(f^*(X) - Y)^2] = \mathbb{E}[\epsilon^2] = \sigma^2$$

Notice: Optimal predictor does not have zero error

$$\mathbb{E}_{D_n}[R(\widehat{f}_n)] = \mathbb{E}_{X,Y,D_n}[(\widehat{f}_n(X) - Y)^2]$$

$D_n$ - training data of size $n$

$$= \mathbb{E}_{X,Y,D_n}\left[(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)] + \mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right]$$

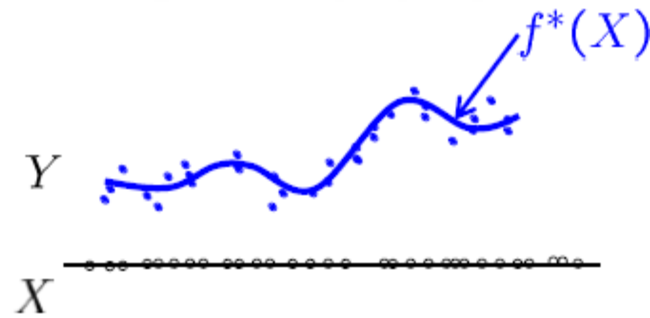
$$= \mathbb{E}_{X,Y,D_n}\Big[(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)])^2 + (\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2$$
$$+2(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)])(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)\Big]$$

$$= \mathbb{E}_{X,Y,D_n}\left[(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)])^2\right] + \mathbb{E}_{X,Y,D_n}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right]$$
$$+\mathbb{E}_{X,Y}\left[2(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - \mathbb{E}_{D_n}[\widehat{f}_n(X)])(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)\right]$$

0

# Bias – Variance Tradeoff: Derivation

Regression: $Y = f^*(X) + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \sigma^2)$




$$R^* = \mathbb{E}_{XY}[(f^*(X) - Y)^2] = \mathbb{E}[\epsilon^2] = \sigma^2$$

Notice: Optimal predictor does not have zero error

$$\mathbb{E}_{D_n}[R(\widehat{f}_n)] = \mathbb{E}_{X,Y,D_n}[(\widehat{f}_n(X) - Y)^2]$$

$D_n$ - training data of size $n$

$$= \mathbb{E}_{X,Y,D_n}\left[(\widehat{f}_n(X) - \mathbb{E}_{D_n}[\widehat{f}_n(X)])^2\right] + \mathbb{E}_{X,Y,D_n}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right]$$

variance – how much does the predictor vary about its mean for different training datasets

Now, lets look at the second term:

$$\mathbb{E}_{X,Y,D_n}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right] = \mathbb{E}_{X,Y}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right]$$

Note: this term doesn't depend on $D_n$

# Bias – Variance Tradeoff: Derivation

$$\mathbb{E}_{X,Y}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - Y)^2\right] = \mathbb{E}_{X,Y}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X) - \epsilon)^2\right]$$

$$= \mathbb{E}_{X,Y}\Big[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))^2 + \epsilon^2$$

$$-2\epsilon(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))\Big]$$

$$= \mathbb{E}_{X,Y}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))^2\right] + \mathbb{E}_{X,Y}\left[\epsilon^2\right]$$

$$-2\mathbb{E}_{X,Y}\left[\epsilon(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))\right]$$

**0** since noise is independent and zero mean

$$= \mathbb{E}_{X,Y}\left[(\mathbb{E}_{D_n}[\widehat{f}_n(X)] - f^*(X))^2\right] + \mathbb{E}_{X,Y}\left[\epsilon^2\right]$$

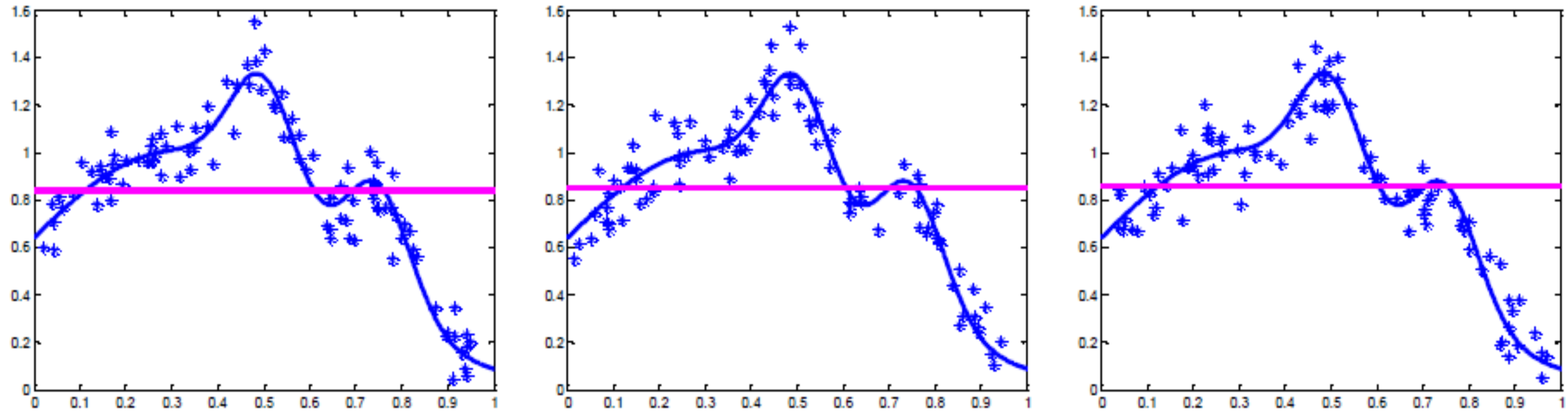
bias^2 – how much does the mean of the predictor differ from the optimal predictor

noise variance

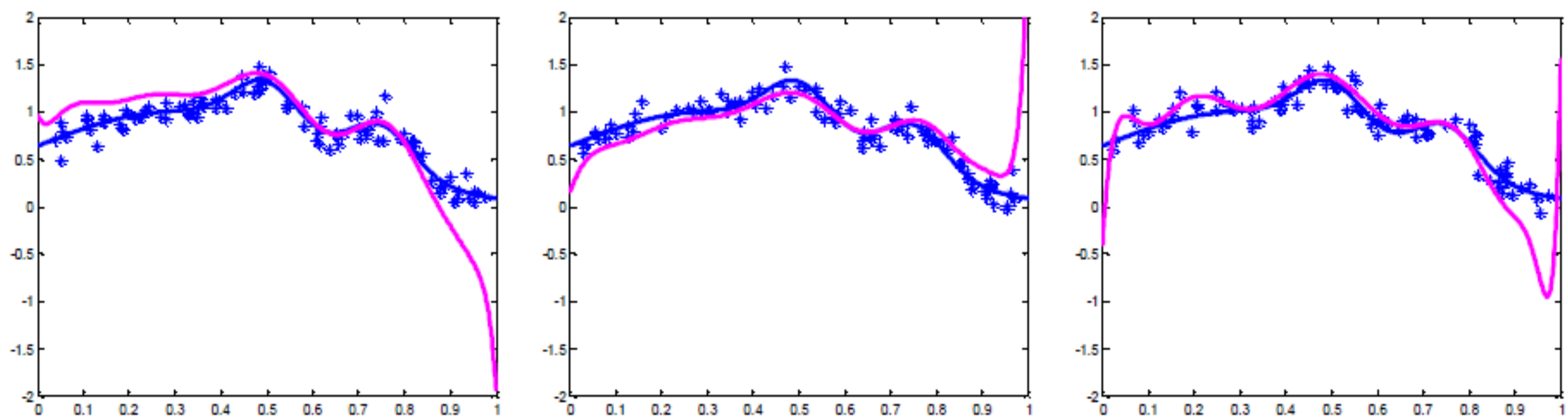
# Bias – Variance Tradeoff

3 Independent training datasets

Large bias, Small variance – poor approximation but robust/stable



Small bias, Large variance – good approximation but instable



# Examples of Model Spaces

Model Spaces with increasing complexity:

- Nearest-Neighbor classifiers with varying neighborhood sizes k = 1,2,3,…
  Small neighborhood => Higher complexity

- Decision Trees with depth k or with k leaves
  Higher depth/ More # leaves => Higher complexity

- Regression with polynomials of order k = 0, 1, 2, …
  Higher degree => Higher complexity

- Kernel Regression with bandwidth h
  Small bandwidth => Higher complexity

**How can we select the right complexity model ?**

# Model Selection

Setup:

Model Classes $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ of increasing complexity $\mathcal{F}_1 \prec \mathcal{F}_2 \prec \ldots$

$$\min_{\lambda} \min_{f \in \mathcal{F}_\lambda} J(f, \lambda)$$

We can select the right complexity model in a data-driven/adaptive way:

- Cross-validation
- Structural Risk Minimization
- Complexity Regularization
- *Information Criteria* - AIC, BIC, Minimum Description Length (MDL)

# Hold-out method

We would like to pick the model that has smallest generalization error.

Can judge generalization error by using an independent sample of data.

Hold – out procedure:

n data points available $D \equiv \{X_i, Y_i\}_{i=1}^n$

1) Split into two sets: Training dataset $D_T = \{X_i, Y_i\}_{i=1}^m$ Validation dataset $D_V = \{X_i, Y_i\}_{i=m+1}^n$ NOT test Data !!

2) Use $D_T$ for training a predictor from each model class:

$$\widehat{f}_\lambda = \arg\min_{f \in \mathcal{F}_\lambda} \widehat{R}_T(f)$$

→ Evaluated on training dataset $D_T$

# Hold-out method

3) Use *Dv* to select the model class which has smallest empirical error on $D_V$

$$\widehat{\lambda} = \arg\min_{\lambda \in \Lambda} \widehat{R}_V(\widehat{f}_\lambda)$$

Evaluated on validation dataset $D_V$

4) Hold-out predictor

$$\widehat{f} = \widehat{f}_{\widehat{\lambda}}$$

Intuition: Small error on one set of data will not imply small error on a randomly sub-sampled second set of data

Ensures method is "stable"

# Hold-out method

Drawbacks:

- May not have enough data to afford setting one subset aside for getting a sense of generalization abilities
- Validation error may be misleading (bad estimate of generalization error) if we get an "unfortunate" split

Limitations of hold-out can be overcome by a family of random sub-sampling methods at the expense of more computation.

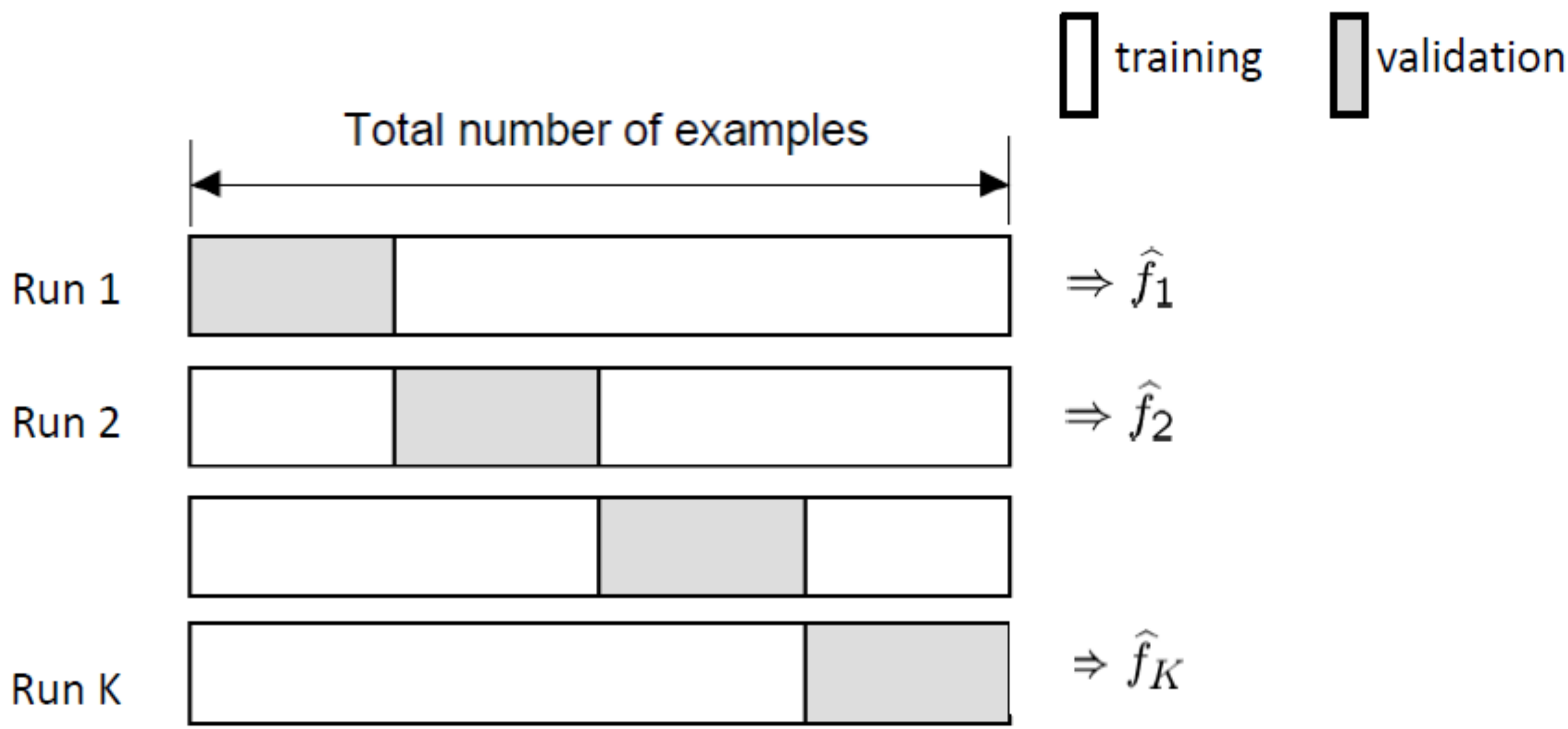
# Cross-validation

## K-fold cross-validation

Create K-fold partition of the dataset.

Form K hold-out predictors, each time using one partition as validation and rest K-1 as training datasets.

Final predictor is average/majority vote over the K hold-out estimates.

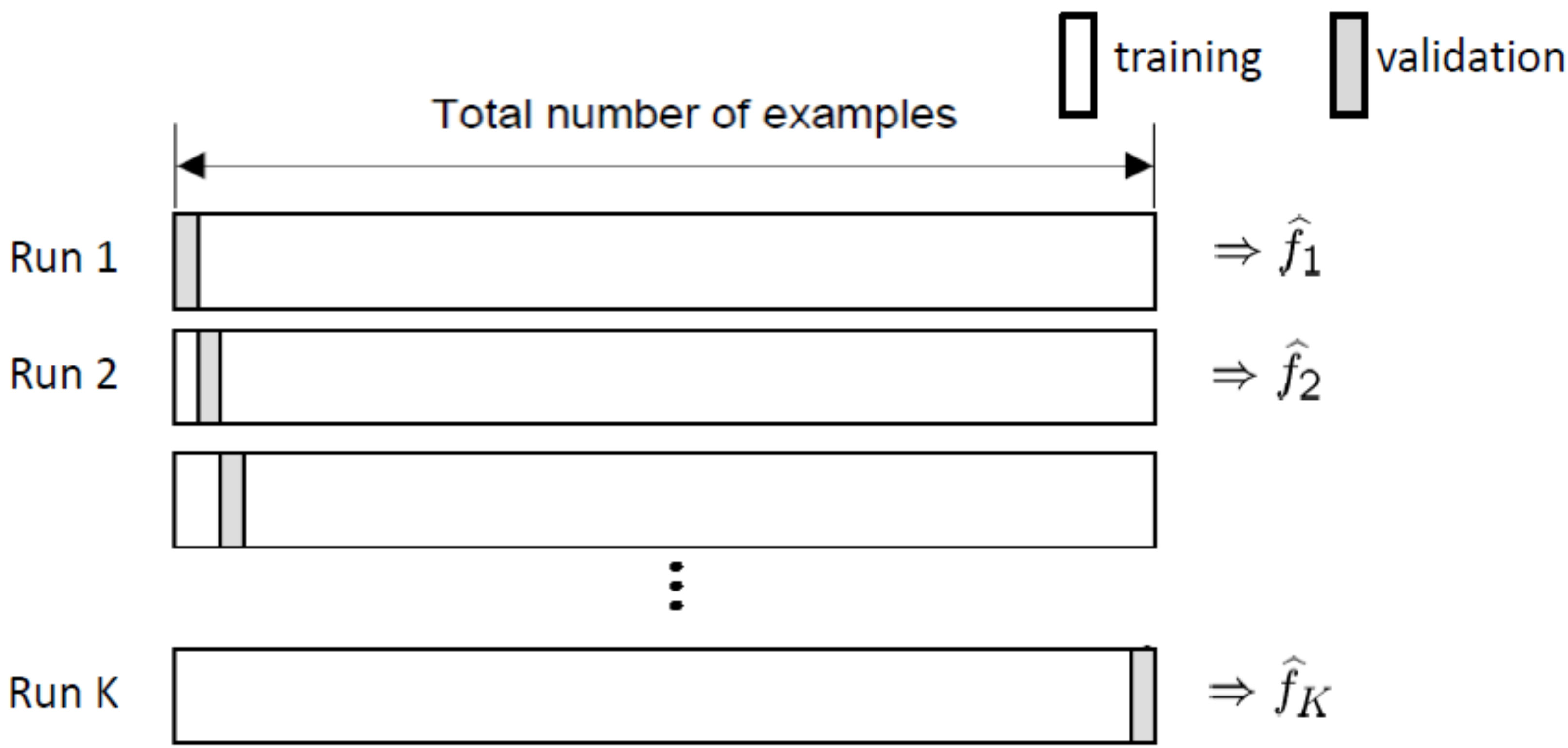


# Cross-validation

## Leave-one-out (LOO) cross-validation

Special case of K-fold with K=n partitions

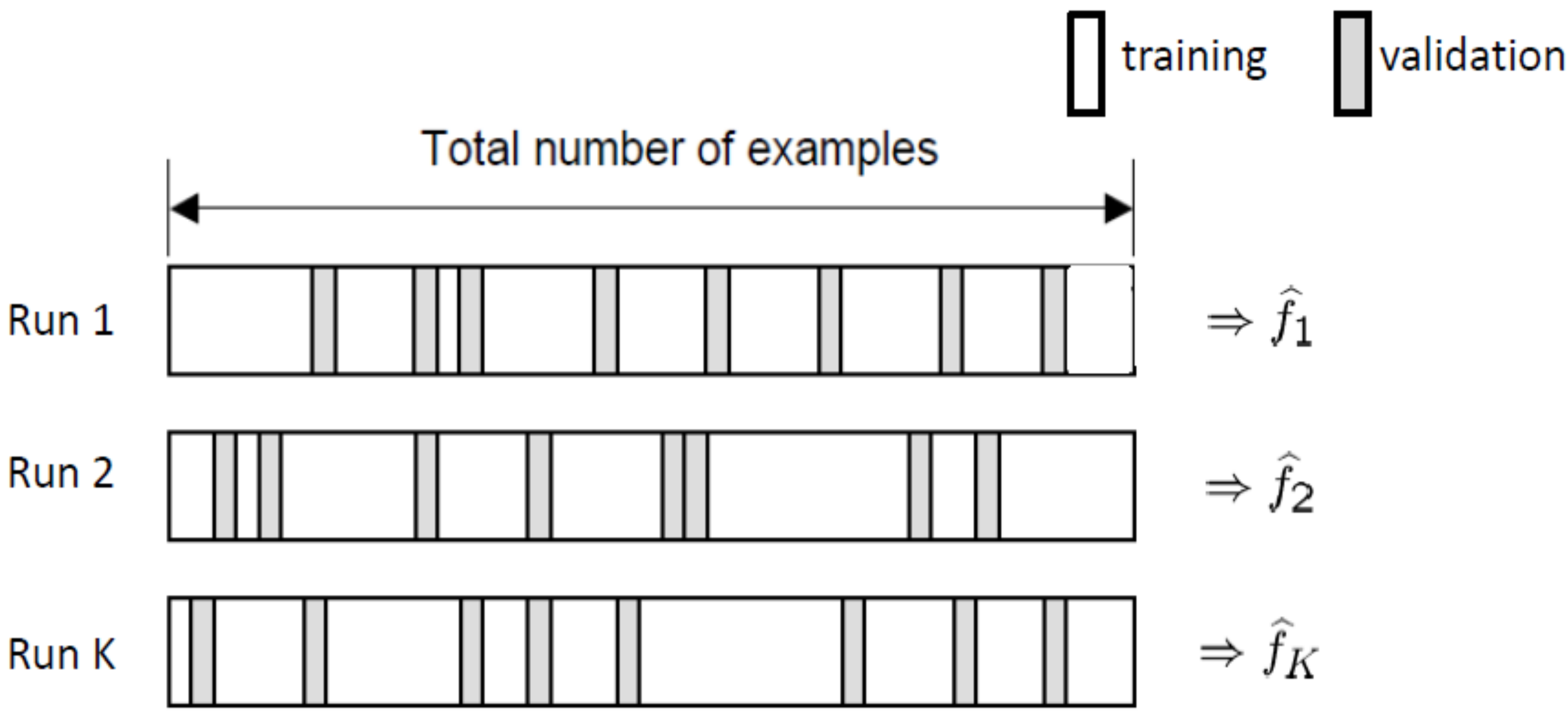
Equivalently, train on n-1 samples and validate on only one sample per run for n runs



# Cross-validation

## Random subsampling

Randomly subsample a fixed fraction $\alpha n$ (0< $\alpha$ <1) of the dataset for validation.
Form hold-out predictor with remaining data as training data.
Repeat K times
Final predictor is average/majority vote over the K hold-out estimates.



# Estimating generalization error

Generalization error $\mathbb{E}_D[R(\widehat{f}_n)]$

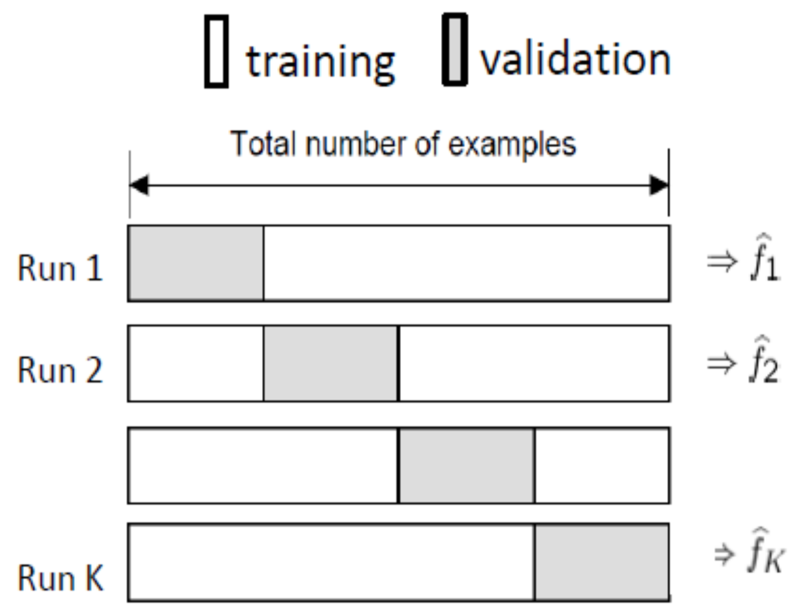
Hold-out ≡ 1-fold: Error estimate = $\widehat{R}_V(\widehat{f}_T)$

K-fold/LOO/random sub-sampling: Error estimate = $\frac{1}{K}\sum_{k=1}^{K}\widehat{R}_{V_k}(\widehat{f}_{T_k})$

We want to estimate the error of a predictor based on n data points.

If K is large (close to n), bias of error estimate is small since each training set has close to n data points.

However, variance of error estimate is high since each validation set has fewer data points and $\widehat{R}_{V_k}$ might deviate a lot from the mean.



# Practical Issues in Cross-validation

How to decide the values for *K* and *α*?

- Large K
  + The bias of the error estimate will be small
  - The variance of the error estimate will be large (few validation pts)
  - The computational time will be very large as well (many experiments)

- Small K
  + The # experiments and, therefore, computation time are reduced
  + The variance of the error estimate will be small (many validation pts)
  - The bias of the error estimate will be large

Common choice: K = 10, $\alpha = 0.1$ ☺

# Structural Risk Minimization

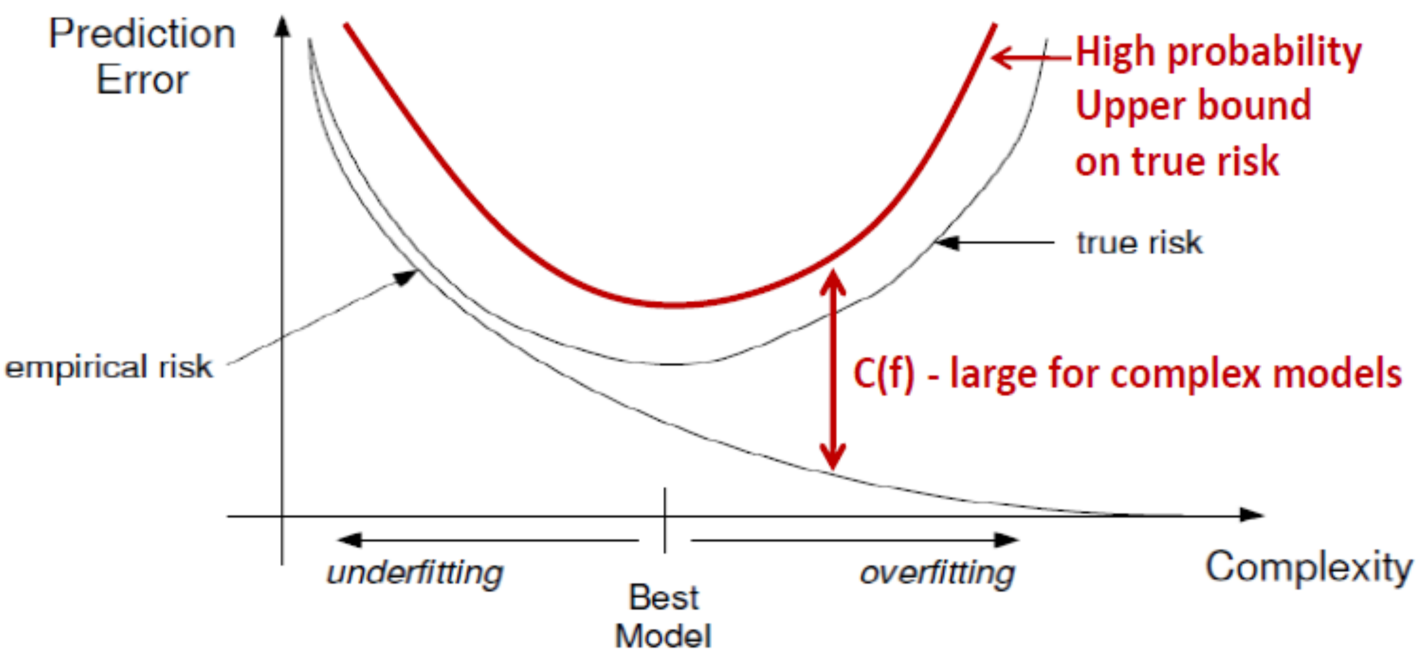
Penalize models using bound on **deviation of true and empirical risks**.

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + C(f) \right\}$$

Bound on deviation from true risk

With high probability, $|R(f) - \widehat{R}_n(f)| \leq C(f) \quad \forall f \in \mathcal{F}$

Concentration bounds (later)



# Structural Risk Minimization

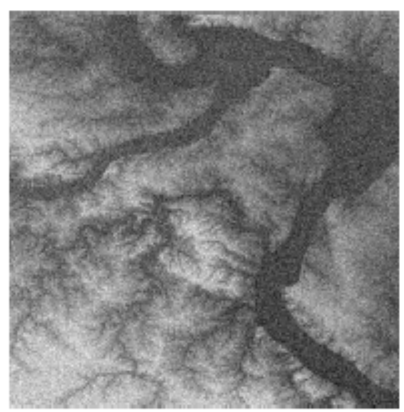
Deviation bounds are typically pretty loose, for small sample sizes. In practice,

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + \lambda C(f) \right\}$$

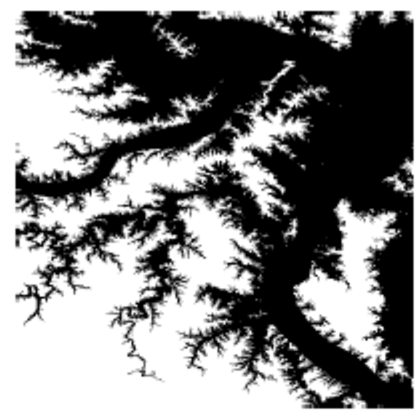
Choose by cross-validation!

Problem: Identify flood plain from noisy satellite images

Noiseless image



Noisy image



True Flood plain
(elevation level > x)

# Structural Risk Minimization

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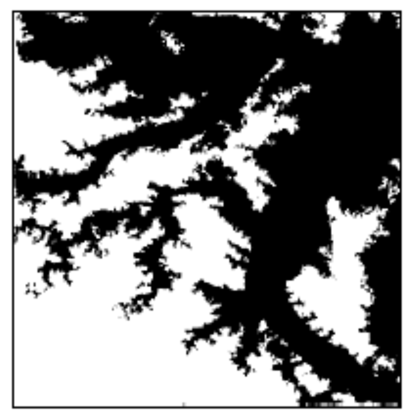
Choose by cross-validation!

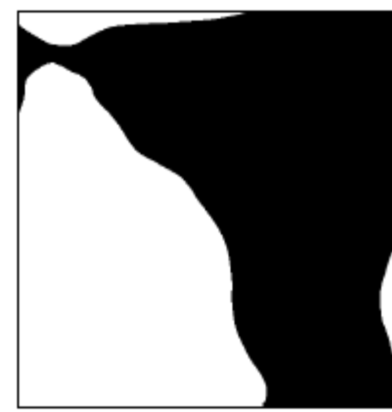
**Problem:** Identify flood plain from noisy satellite images

True Flood plain
(elevation level > x)

Zero penalty



CV penalty



Theoretical penalty

# Occam's Razor

William of Ockham (1285-1349) *Principle of Parsimony:*

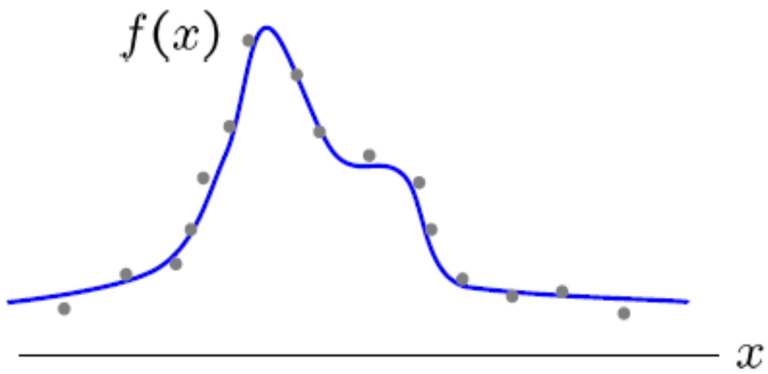
"One should not increase, beyond what is necessary, the number of entities required to explain anything."

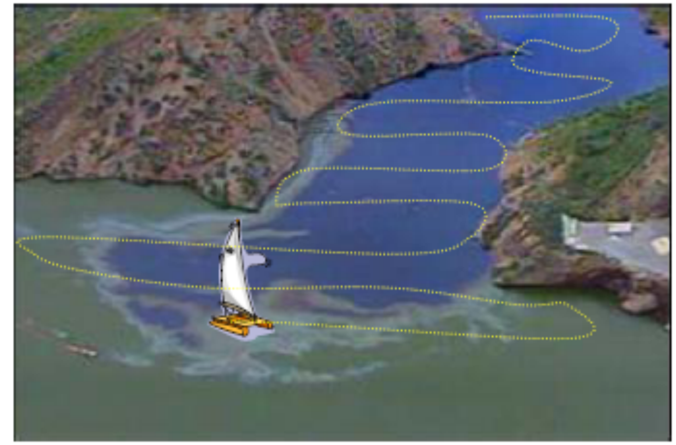
Alternatively, seek the simplest explanation.

Penalize complex models based on

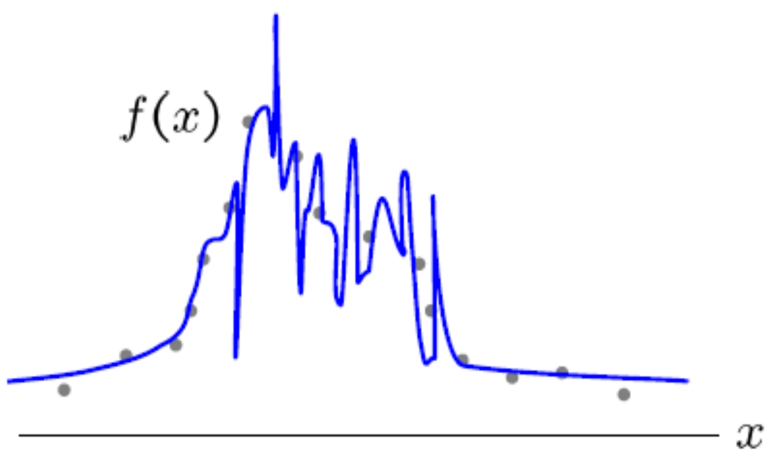
- Prior information (bias)
- Information Criterion (MDL, AIC, BIC)

# Importance of Domain knowledge



Oil Spill Contamination




Distribution of photon arrivals

Compton Gamma-Ray Observatory Burst and Transient Source Experiment (BATSE)

# Complexity Regularization

Penalize complex models using **prior knowledge**.

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + C(f) \right\}$$

Cost of model
(log prior)

Bayesian viewpoint:

prior probability of *f*, p*(f)* ≡ $e^{-C(f)}$

cost is small if *f* is highly probable, cost is large if *f* is improbable

ERM (empirical risk minimization) over a restricted class *F*
≡ uniform prior on *f* ϵ *F*, zero probability for other predictors

$$\widehat{f}_n^L = \arg\min_{f \in \mathcal{F}_L} \widehat{R}_n(f)$$

# Complexity Regularization

Penalize complex models using **prior knowledge**.

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + C(f) \right\}$$

→ Cost of model (log prior)

Examples: MAP estimators

Regularized Linear Regression - Ridge Regression, Lasso

$$\widehat{\theta}_{\mathsf{MAP}} = \arg\max_{\theta} \log p(D|\theta) + \log p(\theta)$$

$$\widehat{\beta}_{\mathsf{MAP}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - X_i\beta)^2 + \lambda \|\beta\|$$

→ Penalize models based on some norm of regression coefficients

How to choose tuning parameter λ? **Cross-validation**

# Information Criteria – AIC, BIC

Penalize complex models based on their **information content**.

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + C(f) \right\}$$

$C(f)$ → # bits needed to describe *f* (description length)

**AIC (Akiake IC)** $C(f)$ = # parameters

Allows # parameters to be infinite as # training data n become large

**BIC (Bayesian IC)** $C(f)$ = # parameters * log n

Penalizes complex models more heavily – limits complexity of models as # training data n become large

# Information Criteria - MDL

Penalize complex models based on their **information content.**

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \left\{ \widehat{R}_n(f) + C(f) \right\}$$

→ # bits needed to describe *f* (description length)

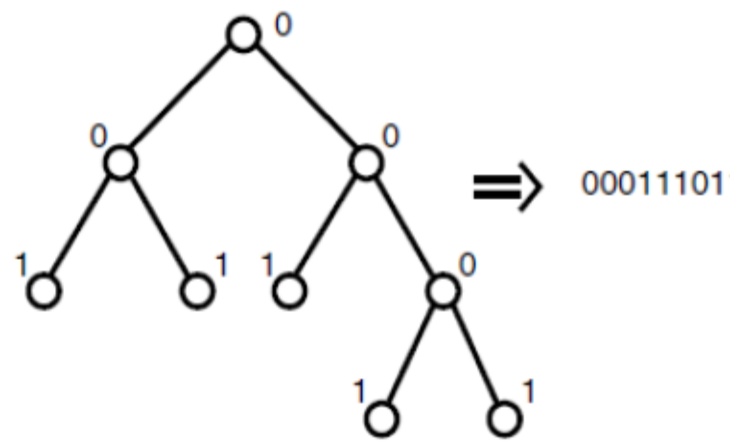
**MDL (Minimum Description Length)**

Example: Binary Decision trees $\mathcal{F}_k^T = \{\text{tree classifiers with } k \text{ leafs}\}$

$$C(f) = 3k - 1 \text{ bits}$$

k leaves => 2k – 1 nodes

2k – 1 bits to encode tree structure
\+ k bits to encode label of each leaf (0/1)




5 leaves => 9 bits to encode structure

# Summary

True and Empirical Risk

Over-fitting

Approx err vs Estimation err, Bias vs Variance tradeoff

Model Selection, Estimating Generalization Error

- Hold-out, K-fold cross-validation
- Structural Risk Minimization
- Complexity Regularization
- Information Criteria – AIC, BIC, MDL