Linear and Integer Programming

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References

- Princeton's class notes on linear programming
- MIT's class notes on linear programming
- Xian Jiaotong University's class notes on linear programming
- Ohio University's class notes
- Rutgers University's class notes

Linear Programming

- Essential tool for optimal allocation of scarce resource among a number of competing activities
- Powerful and general problem-solving method

  - shortest path, max flow, min cost flow

  minimum spanning tree

Why Significant?

- Fast commercial solvers: CPLEX, OSL
- Ranked among most important scientific advances of 20th century
- Also a general tool for attacking NP-hard optimization problems
- Dominates world of industry

  - ex: Delta claims saving $100 million per year using LP

Applications

- Agriculture: diet problem
- Computer science: data mining
- Electrical engineering: VLSI design
- Energy: blending petroleum products
- Environment: water quality management
- Finance: portfolio optimization
- Logistics: supply-chain management
- Management: hotel yield management
- Marketing: direct mail advertising
- manufacturing: production line balancing
- Medicine: radioactive seed placement in cancer treatment
- Operation research: airline crew assignment
- Telecommunication: network design, internet routing
- Sports: scheduling ACC basketball

An Example, Intuitive Understanding of Linear Programming

Brewery Problem: A Toy LP Example

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

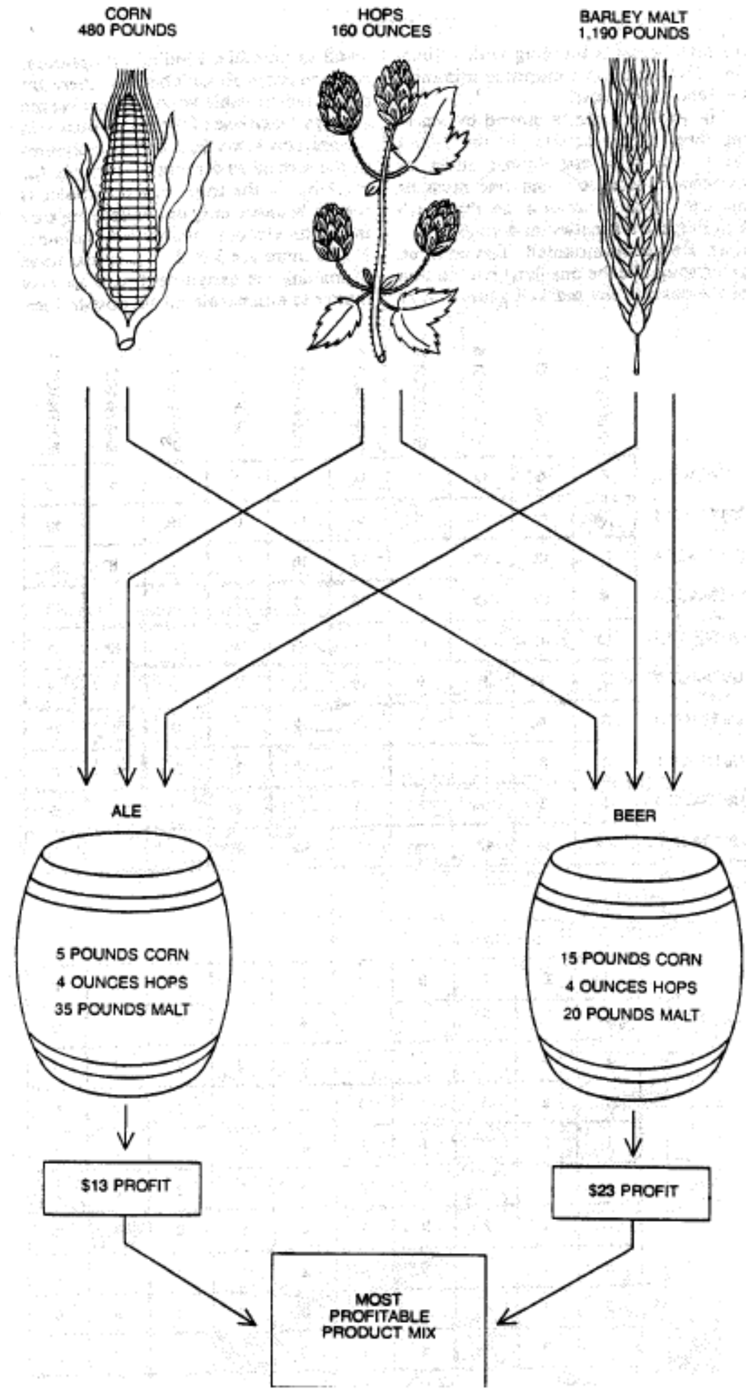
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale	5	4	35	13
Beer	15	4	20	23
Quantity	480	160	1190	

How can brewer maximize profits?

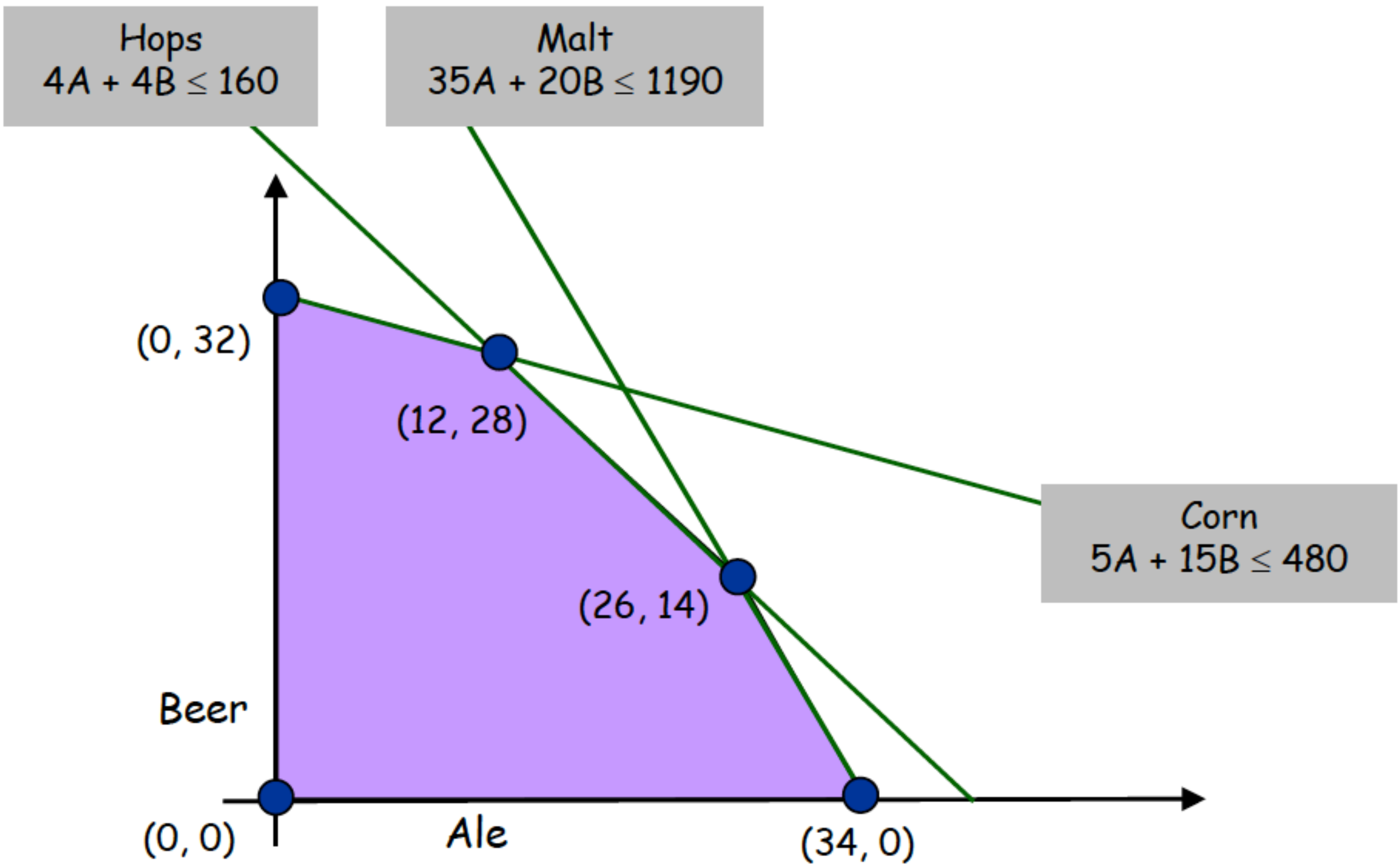
- Devote all resources to ale: 34 barrels of ale $\Rightarrow$ $442.
- Devote all resources to beer: 32 barrels of beer $\Rightarrow$ $736.
- 7.5 barrels of ale, 29.5 barrels of beer $\Rightarrow$ $776.
- 12 barrels of ale, 28 barrels of beer $\Rightarrow$ $800.

Brewery Problem

	Ale		Beer			
max	$13A$	+	$23B$			Profit
s.t.	$5A$	+	$15B$	$\leq$	480	Corn
	$4A$	+	$4B$	$\leq$	160	Hops
	$35A$	+	$20B$	$\leq$	1190	Malt
	A	,	B	$\geq$	0	



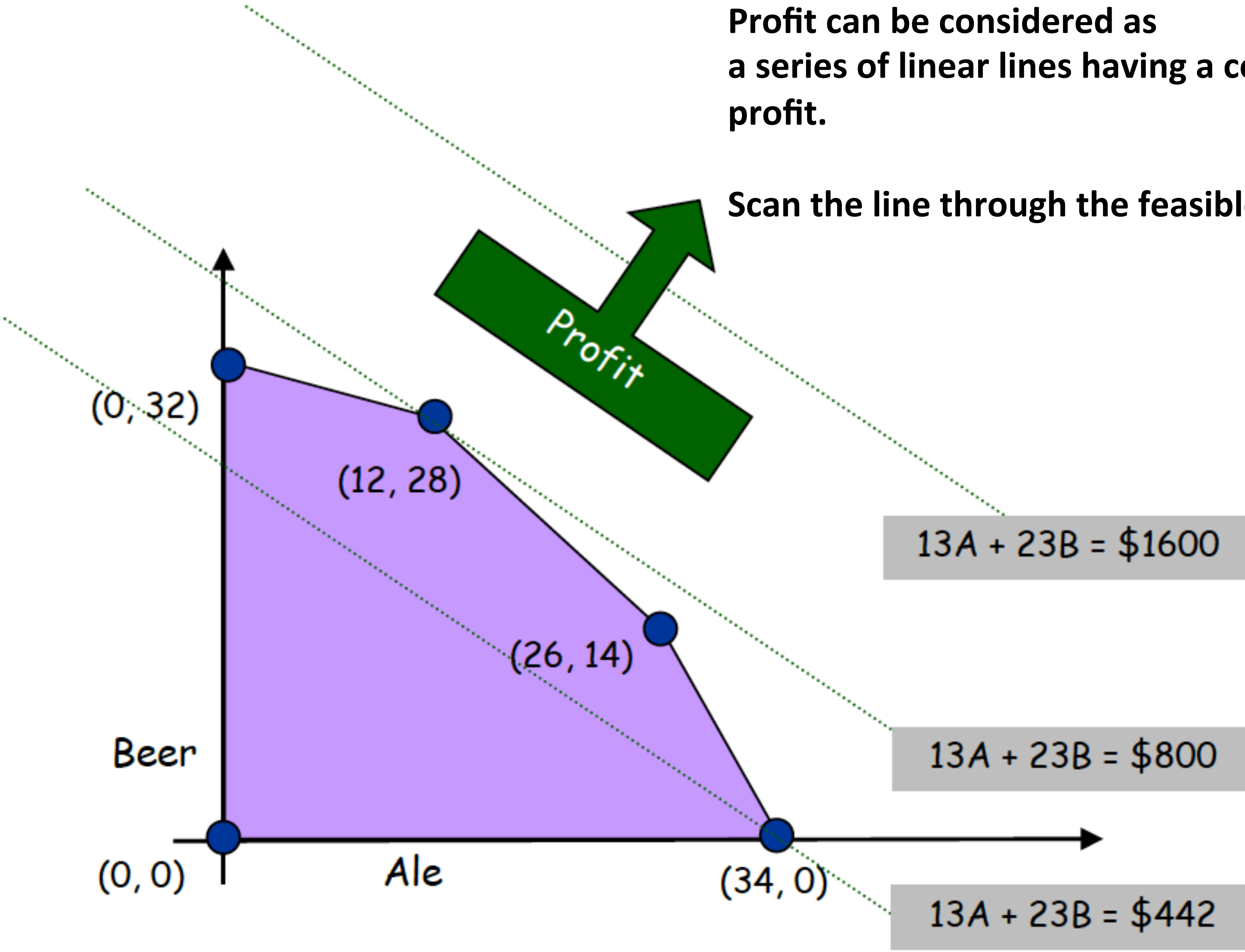
Brewery Problem: Feasible Region



Brewery Problem: Objective Function

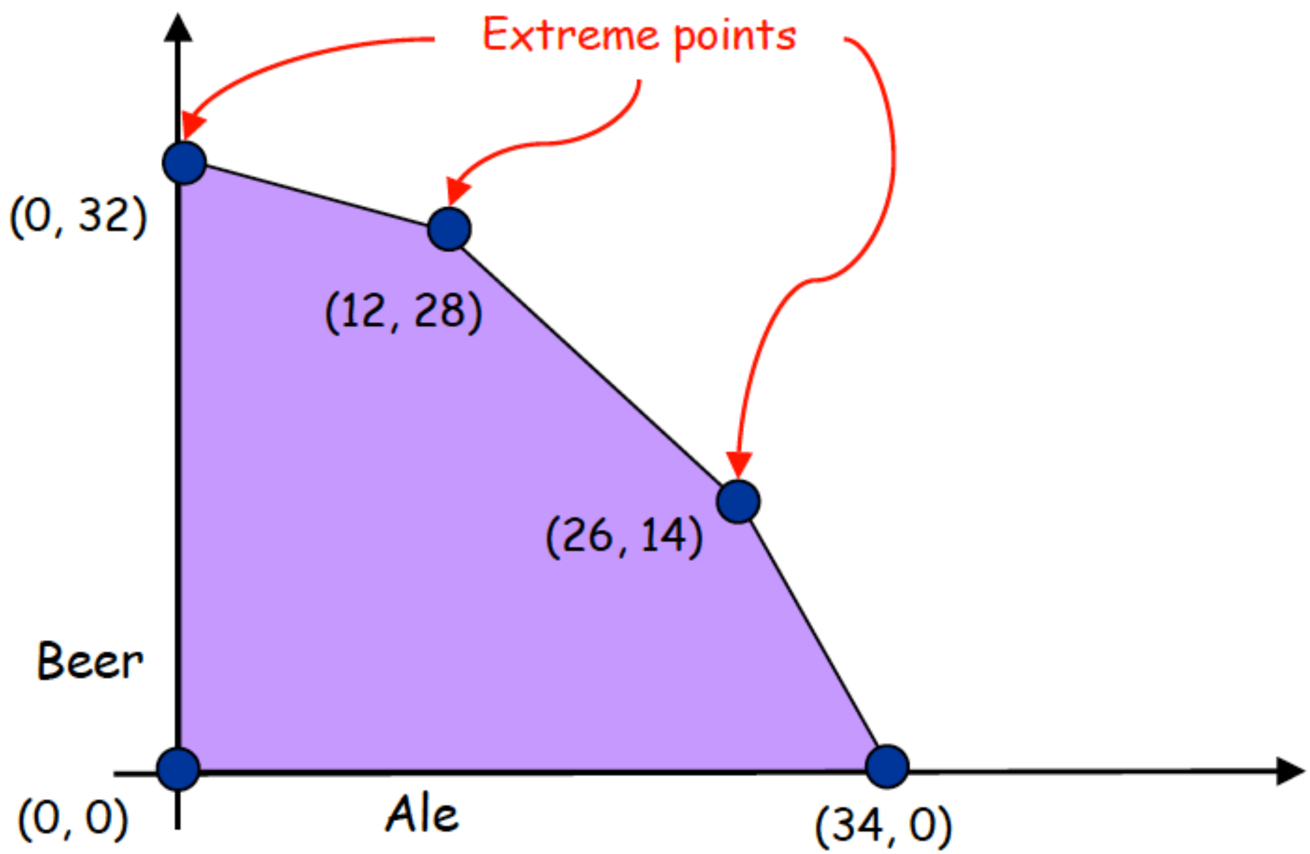
Profit can be considered as a series of linear lines having a certain profit.

Scan the line through the feasible area.



Brewery Problem: Geometry

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at an extreme point.



Standard Form LP

"Standard form" LP.

- Input: real numbers c_j, b_i, a_{ij} .
- Output: real numbers x_j.
- n = # nonnegative variables, m = # constraints.
- Maximize linear objective function subject to linear inequalities.

$$
\begin{aligned}
(P)\ \max\ & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.}\ & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \le i \le m \\
& x_j \ge 0 \quad 1 \le j \le n
\end{aligned}
$$

$$
\begin{aligned}
(P)\ \max\ & c^T x \\
\text{s.t.}\ & Ax = b \\
& x \ge 0
\end{aligned}
$$

Linear. No x^2, xy, $\arccos(x)$, etc.

Programming. Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input.

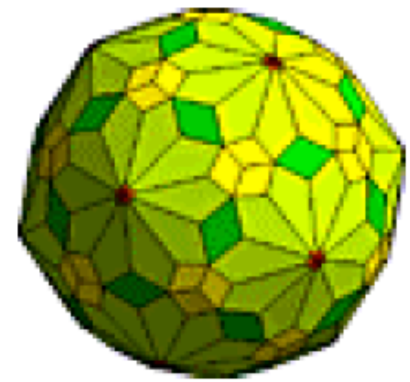
$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Standard form.

- Add slack variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{lrcrcccccccr}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_H & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_M & = & 1190 \\
 & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Geometry

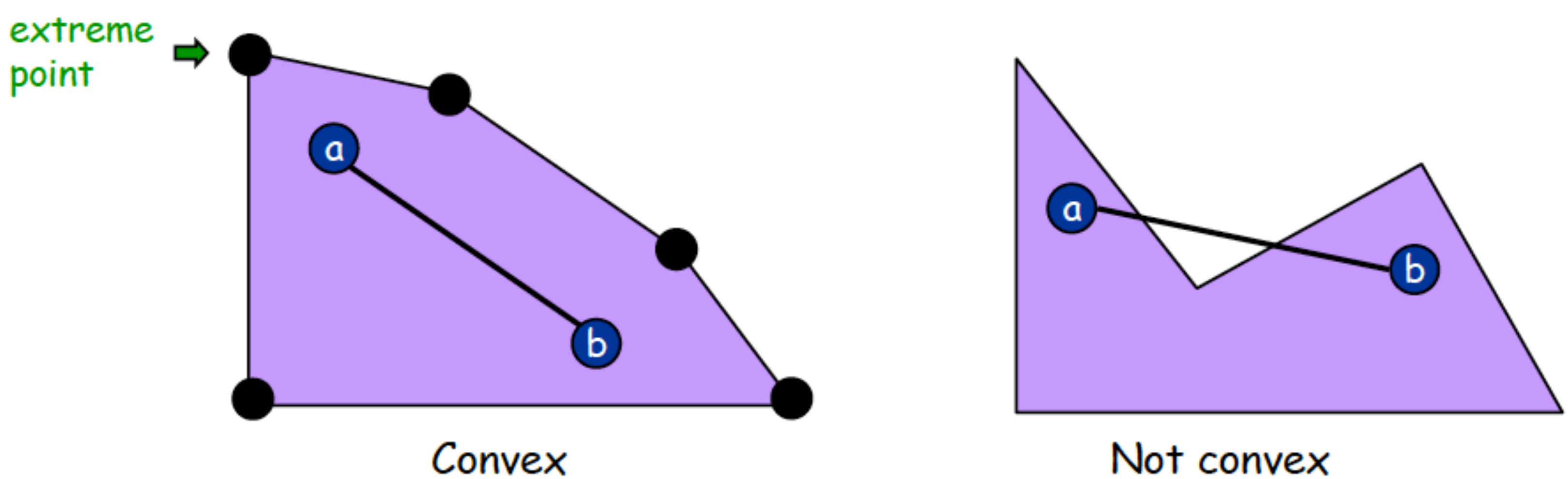


Geometry.

- Inequalities : halfplanes (2D), hyperplanes.
- Bounded feasible region: convex polygon (2D), (convex) polytope.

Convex: if a and b are feasible solutions, then so is (a +b) / 2.

Extreme point: feasible solution x that can't be written as (a + b) / 2 for any two distinct feasible solutions a and b.



Geometry

Extreme point property. If there exists an optimal solution to (P), then there exists one that is an extreme point.

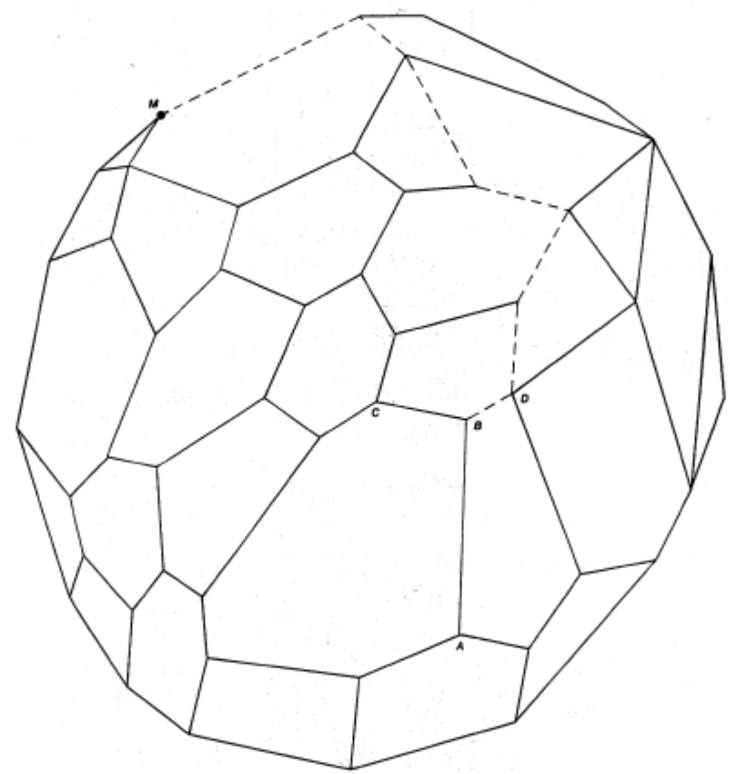
- Only need to consider finitely many possible solutions.

Challenge. Number of extreme points can be exponential!

- Consider n-dimensional hypercube.

Greed. Local optima are global optima.

- Extreme point is optimal if no neighboring extreme point is better.



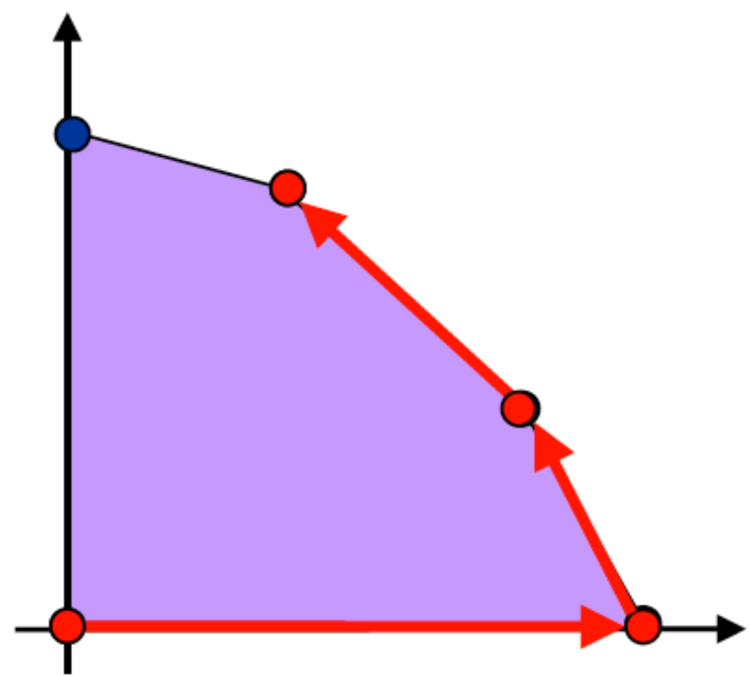
Simplex Algorithm

Simplex algorithm. (George Dantzig, 1947)

- Developed shortly after WWII in response to logistical problems.
- Used for 1948 Berlin airlift.

Generic algorithm.

- Start at some extreme point.
- Pivot from one extreme point to a neighboring one. (never decrease objective function)
- Repeat until optimal.



How to implement? Linear algebra.

Another Example, Formulation, and Proof

Linear Programming

- minimize or maximize a linear objective
- subject to linear equalities and inequalities

Example. Max is in a pie eating contest that lasts 1 hour. Each torte that he eats takes 2 minutes. Each apple pie that he eats takes 3 minutes. He receives 4 points for each torte and 5 points for each pie. What should Max eat so as to get the most points?

Step 1. Determine the decision variables

- Let x be the number of tortes eaten by Max.
- Let y be the number of pies eaten by Max.

Max's linear program

Step 2. Determine the *objective function*

Step 3. Determine the *constraints*

Maximize $z = 4x + 5y$ (objective function)

subject to $2x + 3y \leq 60$ (constraint)

$x \geq 0$; $y \geq 0$ (non-negativity constraints)

A ***feasible solution*** satisfies all of the constraints.

$x = 10$, $y = 10$ is feasible; $x = 10$, $y = 15$ is ***infeasible***.

An ***optimal solution*** is the best feasible solution.

The optimal solution is $x = 30$, $y = 0$.

Terminology

- ***Decision variables:*** e.g., x and y.
  - In general, these are quantities you can control to improve your objective which should completely describe the set of decisions to be made.
- ***Constraints:*** e.g., $2x + 3y \leq 24$, $x \geq 0$, $y \geq 0$
  - Limitations on the values of the decision variables.
- ***Objective Function.*** e.g., $4x + 5y$
  - Value measure used to rank alternatives
  - Seek to maximize or minimize this objective
  - examples: maximize NPV, minimize cost

Linear Programs

- A *linear function* is a function of the form:

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$= \sum_{i=1 \text{ to } n} c_ix_i$$

e.g., $3x_1 + 4x_2 - 3x_4$.

- A mathematical program is a *linear program (LP)* if the objective is a linear function and the constraints are linear equalities or inequalities.

e.g., $3x_1 + 4x_2 - 3x_4 \geq 7$

$x_1 - 2x_5 = 7$

- Typically, an LP has non-negativity constraints.

Max's linear program

Step 2. Determine the *objective function*

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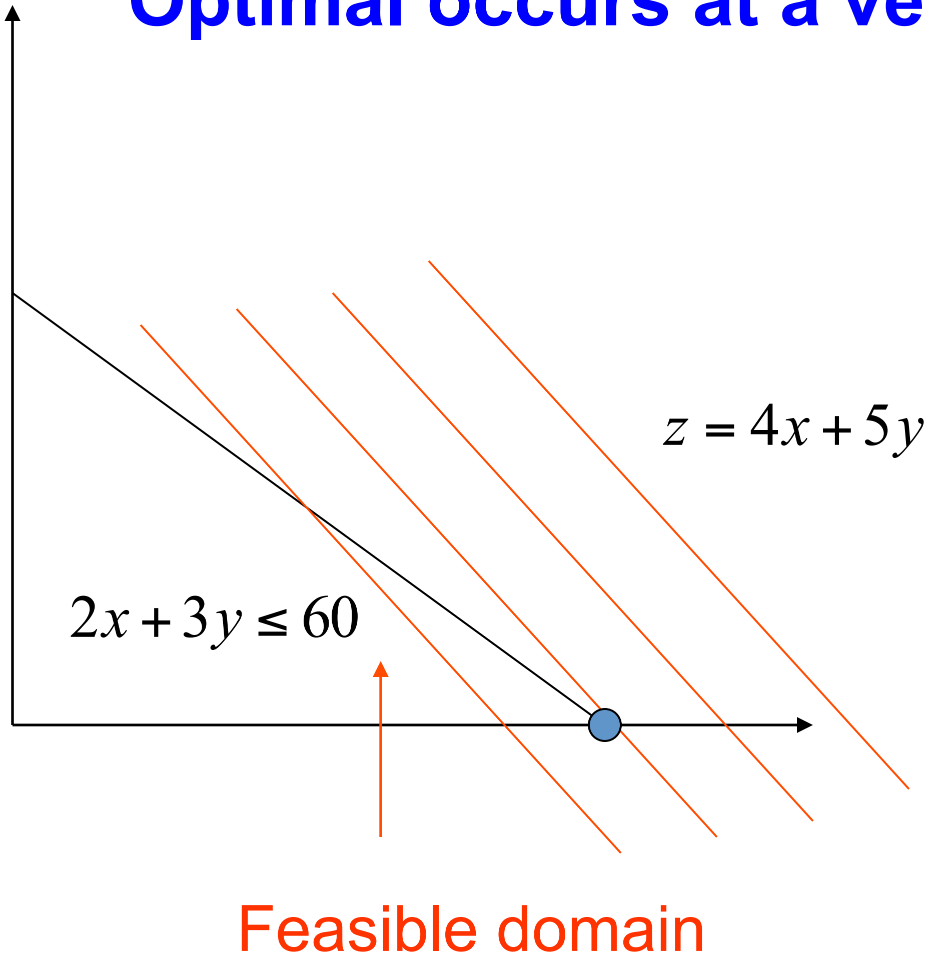
Slack Form

$$\min_{ax} \; z = -4x - 5y$$
$$\text{s.t. } 2x + 3y + w = 60$$
$$x \geq 0, y \geq 9, w \geq 0.$$

$$\min_{x} \; z = cx$$
$$\text{s.t. } \quad Ax = b$$
$$x \geq 0.$$
$$\text{rank}(A) = m.$$

Optimal occurs at a vertex!!!



What' s a vertex?

A point x in a polyhadren Ω is called a vertex if

$$x = \frac{1}{2}(y+z), y, z \in \Omega \quad \Longrightarrow \quad x = y = z.$$

Fundamental Theorem

Let $\Omega = \{x \mid Ax = b,\ x \geq 0\}$.

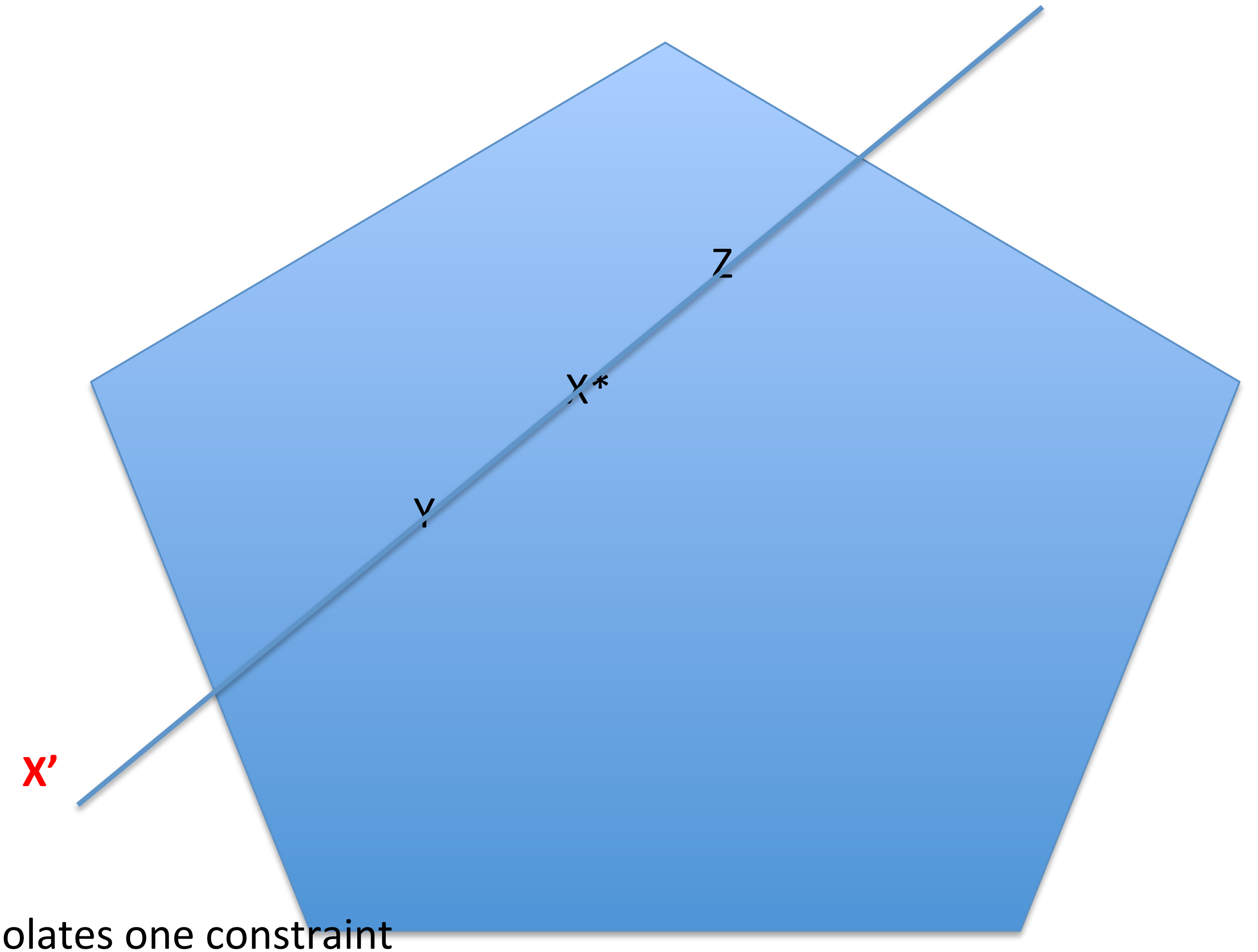
If min cx over $x \in \Omega$ has an optimal solution,

then it can be found in vertices of Ω.

Proof.

Consider an optimal solution x^* with maximum number of zero components among all optimal solutions.

We will show that x^* is a vertex of Ω. By contradition, suppose x^* is not, that is, there exist $y, z \in \Omega$ such that $x^* = (y+z)/2$ and x^*, y, z are distinct. Since $cx^* \le cy$, $cx^* \le cz$, and $cx^* = (cy+cz)/2$, we must have $cx^* = cy = cz$. This means that y and z are also optimal solutions. It follows that all feasible points on line $x^*+\alpha(y-x^*)$ are optimal solutions. However, Ω does not contain any infinite line. Thus, the line must have a point x' not in Ω, that is, x' violates at least one constraint.



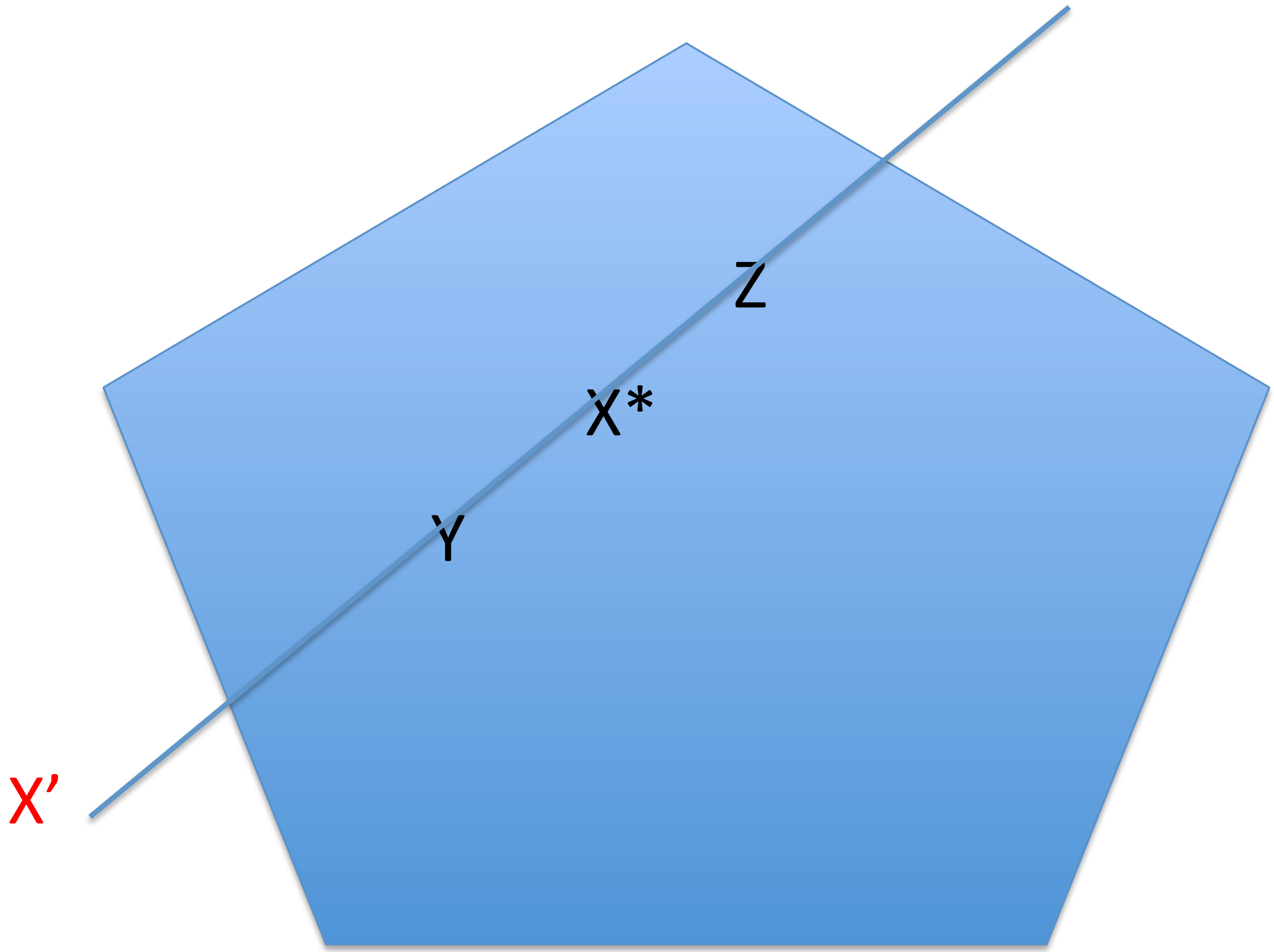
X' violates one constraint

Proof (cont’s).

Note that for any α, $A(x^*+\alpha(y\text{-}x^*)) = b$.

Thus, x' cannot violate constraint $Ax = b$. Moreover, for $x_i^* = 0$, since $x_i^* = (y_i + z_i)/2$ and $y_i, z_i \geq 0$, we must have $z_i = y_i = x_i^* = 0$. Therefore, the ith component of $x^*+\alpha(y\text{-}x^*)$ is equal to 0 for any α. This means that x' cannot violate constraint $x_i \geq 0$. Hence, x' must violate a constraint $x_j \geq 0$ for some j with $x_j^* > 0$.

Now, we can easily find an optimal solution between x' and x^*, which has one more zero-component than x^*, a contradiction.



X’ has fewer or equal number of 0 terms as
X*. X’ has one negative component, whose value in X* is positive.
As we move from X’ to X* into the feasible region, the component will become 0

Solution Approach

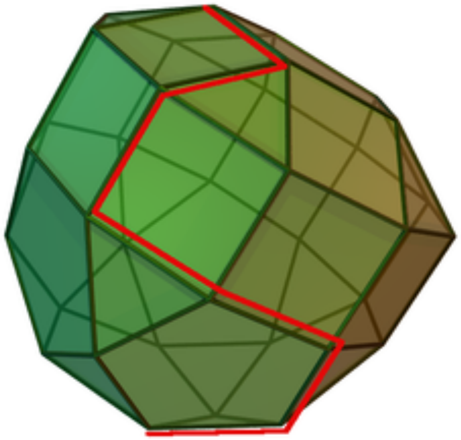
- Find a corner point
  - An "initial feasible solution"
- Proceed to improved corner points
- Stop when no further improvements are possible

Solution Calculations

- To find a corner point
  - it is necessary to solve system of constraint equations
  - from linear algebra, this requires working with matrix of constraint equations, specifically, manipulating the “determinants”
  - Amount of effort set by number of constraints
- Thus, number of constraints defines amount of effort
- This is why LP can handle many more decision variables than constraints

Solution Methods

- Simplex
  - The textbook method
  - For step 2, select improved corners
    - *Always goes to best corner*
    - *Searches until no further improvement possible*
  - Inefficient for real problems
  - Not used in practice

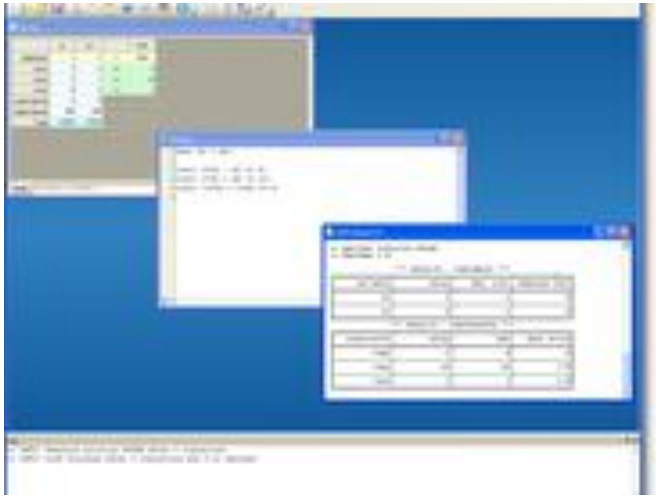


Polyhedron of simplex algorithm in 3D

- Practical methods - many exist - often proprietary
  - Step 2 takes many forms
  - Each best for different cases
  - Very great efficiency possible
  - A real art!

Demo of MIT Linear Programming Solver:

http://sourceforge.net/projects/lipside/



LP Duality: Economic Interpretation

Brewer's problem: find optimal mix of beer and ale to maximize profits.

$$
\begin{aligned}
\text{(P)} \quad \max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \le 480 \\
& 4A + 4B \le 160 \\
& 35A + 20B \le 1190 \\
& A, \; B \ge 0
\end{aligned}
$$

$A^* = 12$
$B^* = 28$
$OPT = 800$

Entrepreneur's problem: Buy individual resources from brewer at minimum cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$
\begin{aligned}
\text{(D)} \quad \min \quad & 480C + 160H + 1190M \\
\text{s.t.} \quad & 5C + 4H + 35M \ge 13 \\
& 15C + 4H + 20M \ge 23 \\
& C, \; H, \; M \ge 0
\end{aligned}
$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

LP Duality

Primal and dual LPs. Given real numbers a_{ij}, b_i, c_j, find real numbers x_i, y_j that optimize (P) and (D).

$$
\begin{aligned}
\text{(P)} \quad \max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m \\
& x_j \ge 0 \quad 1 \le j \le n
\end{aligned}
$$

$$
\begin{aligned}
\text{(D)} \quad \min \quad & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad 1 \le j \le n \\
& y_i \ge 0 \quad 1 \le i \le m
\end{aligned}
$$

Duality Theorem (Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947).
If (P) and (D) have feasible solutions, then max = min.

- Special case: max-flow min-cut theorem.
- Sensitivity analysis.

LP Duality: Economic Interpretation

Sensitivity analysis.

- Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
- A. corn $1, hops $2, malt $0.

- Q. New product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
- A. Breakeven: 2 ($1) + 5 ($2) + 24 (0$) = $12 / barrel.

How do I compute marginal prices (dual variables)?

- Simplex solves primal and dual simultaneously.
- Top row of final simplex tableaux provides optimal dual solution!

History

1939. Production, planning. (Kantorovich, USSR)

- Propaganda to make paper more palatable to communist censors.

> "I want to emphasize again that the greater part of the problems of which I shall speak, relating to the organization and planning of production, are connected specifically with the Soviet system of economy and in the majority of cases do not arise in the economy of a capitalist society."
>
> "the majority of enterprises work at half capacity. There the choice of output is determined not by the plan but by the interests and profits of individual capitalists."

USSR

USA

- Kantorovich awarded 1975 Nobel prize in Economics for contributions to the theory of optimum allocation of resources.
- Staple in MBA curriculum.
- Used by most large companies and other profit maximizers.

History

1939. Production, planning. (Kantorovich)
1947. Simplex algorithm. (Dantzig)
1950. Applications in many fields.
1979. Ellipsoid algorithm. (Khachian)
1984. Projective scaling algorithm. (Karmarkar)
1990. Interior point methods.

Current research.

- Approximation algorithms.
- Software for large scale optimization.
- Interior point variants.

Integer Programming

- Integer programming is a solution method for many discrete optimization problems
- Programming = Planning in this context
- Origins go back to military logistics in WWII (1940s).
- In a survey of Fortune 500 firms, 85% of those responding said that they had used linear or integer programming.
- Why is it so popular?
  - Many different real-life situations can be modeled as integer programs (IPs).
  - There are efficient algorithms to solve IPs.

Standard form of integer program (IP)

maximize $c_1x_1+c_2x_2+\ldots+c_nx_n$ (objective function)

subject to

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n \le b_1$ (functional constraints)

$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n \le b_2$

....

$a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n \le b_m$

$x_1, x_2, \ldots, x_n \in \mathbf{Z}_+$ (set constraints)

Standard form of integer program (IP)

- In vector form:

maximize $\mathbf{c}\mathbf{x}$ (objective function)

subject to $A\mathbf{x} \leq \mathbf{b}$ (functional constraints)

$\mathbf{x} \in \mathbf{Z}_{+}^{n}$ (set constraints)

Input for IP: $1\times n$ vector $\mathbf{c}$, $m\times n$ matrice A, $m\times 1$ vector $\mathbf{b}$.

Output of IP: $n\times 1$ integer vector $\mathbf{x}$.

- *Note*: More often, we will consider

mixed integer programs (MIP),

that is, some variables are integer,

the others are continuous.

Example of Integer Program (Production Planning-Furniture Manufacturer)

- Technological data:

  Production of **1 table** requires 5 ft pine, 2 ft oak, 3 hrs labor

  1 chair requires 1 ft pine, 3 ft oak, 2 hrs labor

  1 desk requires 9 ft pine, 4 ft oak, 5 hrs labor

- Capacities for 1 week: 1500 ft pine, 1000 ft oak,

  20 employees (each works 40 hrs).

- Market data:

	profit	**demand**
table	$12/unit	40
chair	$5/unit	130
desk	$15/unit	30

- **Goal:** Find a production schedule for 1 week that will maximize the profit.

Production Planning-Furniture Manufacturer: modeling the problem as integer program

The goal can be achieved

by making appropriate decisions.

First define decision variables:

Let x_t be the number of tables to be produced;

x_c be the number of chairs to be produced;

x_d be the number of desks to be produced.

(Always define decision variables properly!)

Production Planning-Furniture Manufacturer: modeling the problem as integer program

- Objective is to maximize profit:

  $\max\ 12x_t + 5x_c + 15x_d$

- Functional Constraints

  capacity constraints:

  pine: $5x_t + 1x_c + 9x_d \leq 1500$

  oak: $2x_t + 3x_c + 4x_d \leq 1000$

  labor: $3x_t + 2x_c + 5x_d \leq 800$

  market demand constraints:

  tables: $x_t \geq 40$

  chairs: $x_c \geq 130$

  desks: $x_d \geq 30$

- Set Constraints

  $x_t, x_c, x_d \in \mathbf{Z}_+$

Solutions to integer programs

- A *solution* is an assignment of values to variables.
- A *feasible solution* is an assignment of values to variables such that all the constraints are satisfied.
- The *objective function value* of a solution is obtained by evaluating the objective function at the given point.
- An *optimal solution* (assuming maximization) is one whose objective function value is greater than or equal to that of all other feasible solutions.
- Integer program is NP complete
- There are efficient algorithms for finding the optimal solutions of an integer program based on LP relaxation.

Next: IP modeling techniques

Modeling techniques:

- Using binary variables
- Restrictions on number of options
- Contingent decisions
- Variables with k possible values

Applications:

- Facility Location Problem
- Knapsack Problem

Example of IP: Facility Location

- A company is thinking about building new facilities in LA and SF.
- Relevant data:

	capital needed	expected profit
1. factory in LA	$6M	$9M
2. factory in SF	$3M	$5M
3. warehouse in LA	$5M	$6M
4. warehouse in SF	$2M	$4M

Total capital available for investment: $10M

- Question: Which facilities should be built to maximize the total profit?

Example of IP: Facility Location

- Define decision variables (i = 1, 2, 3, 4):

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is built} \\ 0 & \text{if not} \end{cases}$$

- Then the total expected benefit: $9x_1+5x_2+6x_3+4x_4$

  the total capital needed: $6x_1+3x_2+5x_3+2x_4$

➢ Summarizing, the IP model is:

$$\begin{aligned} \max\ & 9x_1+5x_2+6x_3+4x_4 \\ \text{s.t. } & 6x_1+3x_2+5x_3+2x_4 \le 10 \\ & x_1, x_2, x_3, x_4 \text{ binary} \quad (\textit{i.e.}, x_i \in \{0,1\}) \end{aligned}$$

Knapsack problem

Any IP, which has only one constraint, is referred to as a **knapsack problem** .

- n items to be packed in a knapsack.
- The knapsack can hold up to W lb of items.
- Each item has weight w_i lb and benefit b_i .
- Goal: Pack the knapsack such that the total benefit is maximized.

IP model for Knapsack problem

- Define decision variables (i = 1, …, n):

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is packed} \\ 0 & \text{if not} \end{cases}$$

- Then the total benefit: $\sum_{i=1}^{n} b_i x_i$

  the total weight: $\sum_{i=1}^{n} w_i x_i$

➢ Summarizing, the IP model is:

max $\sum_{i=1}^{n} b_i x_i$

s.t. $\sum_{i=1}^{n} w_i x_i \le W$

x_i binary (i = 1, …, n)

Connection between the problems

- *Note*: The version of the facility location problem is a special case of the knapsack problem.

- Important modeling skill:
  - – Suppose you know how to model Problem $A_1,\ldots,A_p$;
  - – You need to solve Problem B;
  - – Notice the similarities between Problems A_i and B;
  - – Build a model for Problem B, using the model for Problem A_i as a prototype.

The Facility Location Problem: adding new requirements

- Extra requirement:

  build *at most one* of the two warehouses.

  The corresponding constraint is:

  $x_3 + x_4 \leq 1$

- Extra requirement:

  build *at least one* of the two factories.

  The corresponding constraint is:

  $x_1 + x_2 \geq 1$

Modeling Technique: Restrictions on the number of options

- *Restrictions:* At least p and at most q of the options can be chosen.
- The corresponding constraints are:

$$\sum_{i=1}^{n} x_i \geq p \qquad \sum_{i=1}^{n} x_i \leq q$$

Modeling Technique: Contingent Decisions

Back to the facility location problem.

- *Requirement:* Can't build a warehouse *unless* there is a factory in the city.

  The corresponding constraints are:

  $x_3 \leq x_1$ (LA) $x_4 \leq x_2$ (SF)

- *Requirement:* Can't select option 3 *unless* at least one of options 1 and 2 is selected.

  The constraint: $x_3 \leq x_1 + x_2$

- *Requirement:* Can't select option 4 *unless* at least two of options 1, 2 and 3 are selected.

  The constraint: $2x_4 \leq x_1 + x_2 + x_3$

Modeling Technique: Variables with k possible values

- Suppose variable y should take one of the values $d_1, d_2, \ldots, d_k$.
- How to achieve that in the model?
- Introduce new decision variables. For $i=1,\ldots,k$,

$$x_i = \begin{cases} 1 & \text{if y takes value } d_i \\ 0 & \text{otherwise} \end{cases}$$

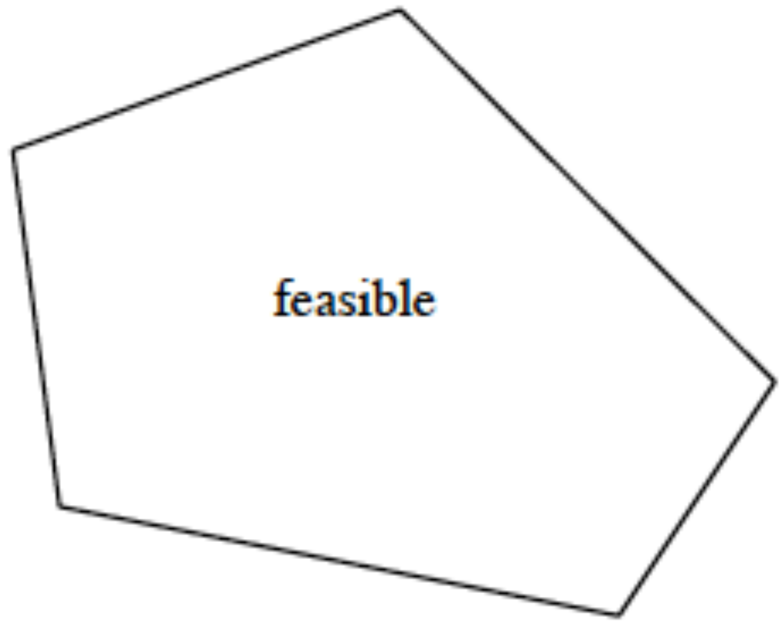
- Then we need the following constraints.

$$\sum_{i=1}^{k} x_i = 1 \quad (y \text{ can take only one value})$$

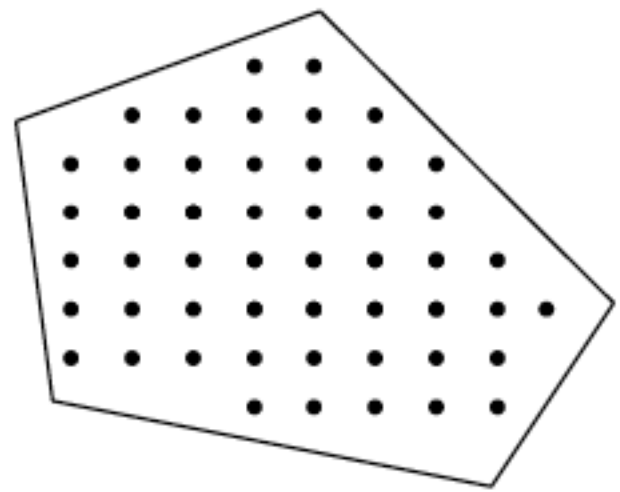
$$y = \sum_{i=1}^{k} d_i x_i \quad (y \text{ should take value } d_i \text{ if } x_i = 1)$$

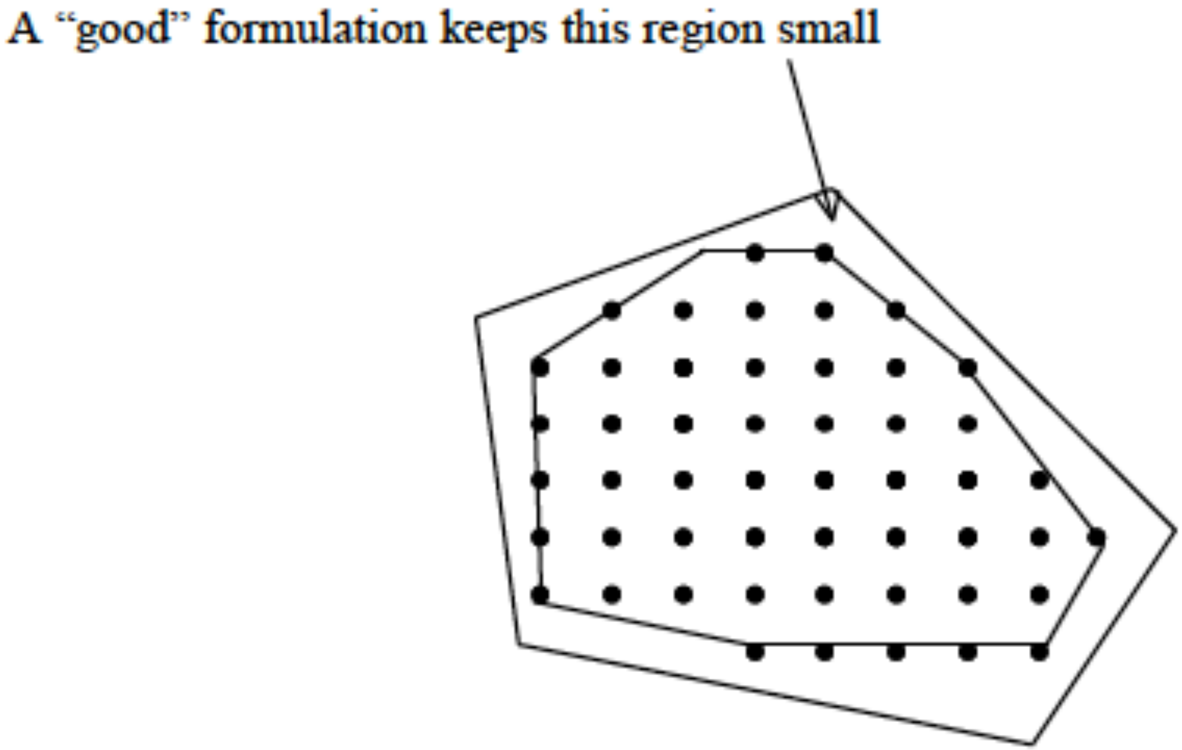
Intersection of all the linear (in)equalities form a convex polytope

- For simplicity, we'll always assume polytope is bounded



Feasible integer points form a lattice inside the LP polytope

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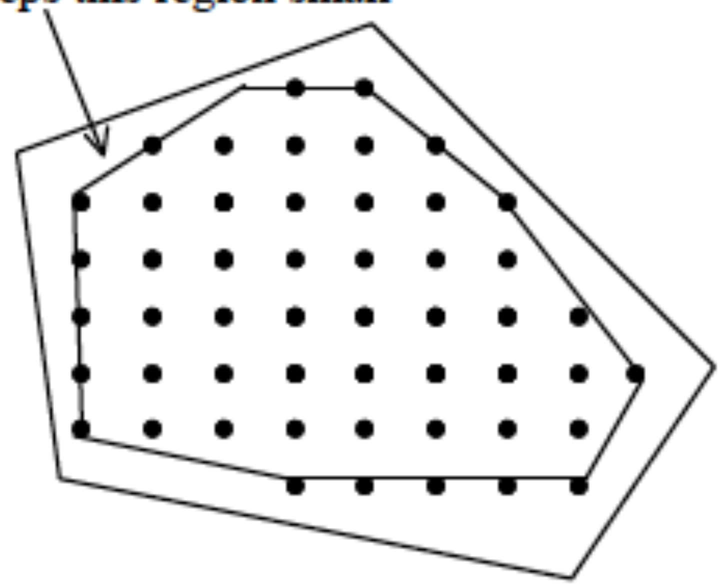


One measure of this is the **Integrality Gap**:

Integrality gap = $\max_{\text{instances } I}(IP(I))/(LP(I))$

- ***Rounding non-integer solution values up*** to the nearest integer value can result in an ***infeasible solution*.**

- A ***feasible solution is ensured by rounding down*** non-integer solution values but may result in a less than optimal ***(sub-optimal) solution.***

Integer Programming Example
Graphical Solution of Machine Shop Model

Maximize $Z = \$100x_1 + \$150x_2$

subject to:

$8{,}000x_1 + 4{,}000x_2 \leq \$40{,}000$

$15x_1 + 30x_2 \leq 200 \text{ ft}^2$

$x_1, x_2 \geq 0$ and integer

Optimal Solution:

$Z = \$1{,}055.56$

$x_1 = 2.22$ presses

$x_2 = 5.55$ lathes

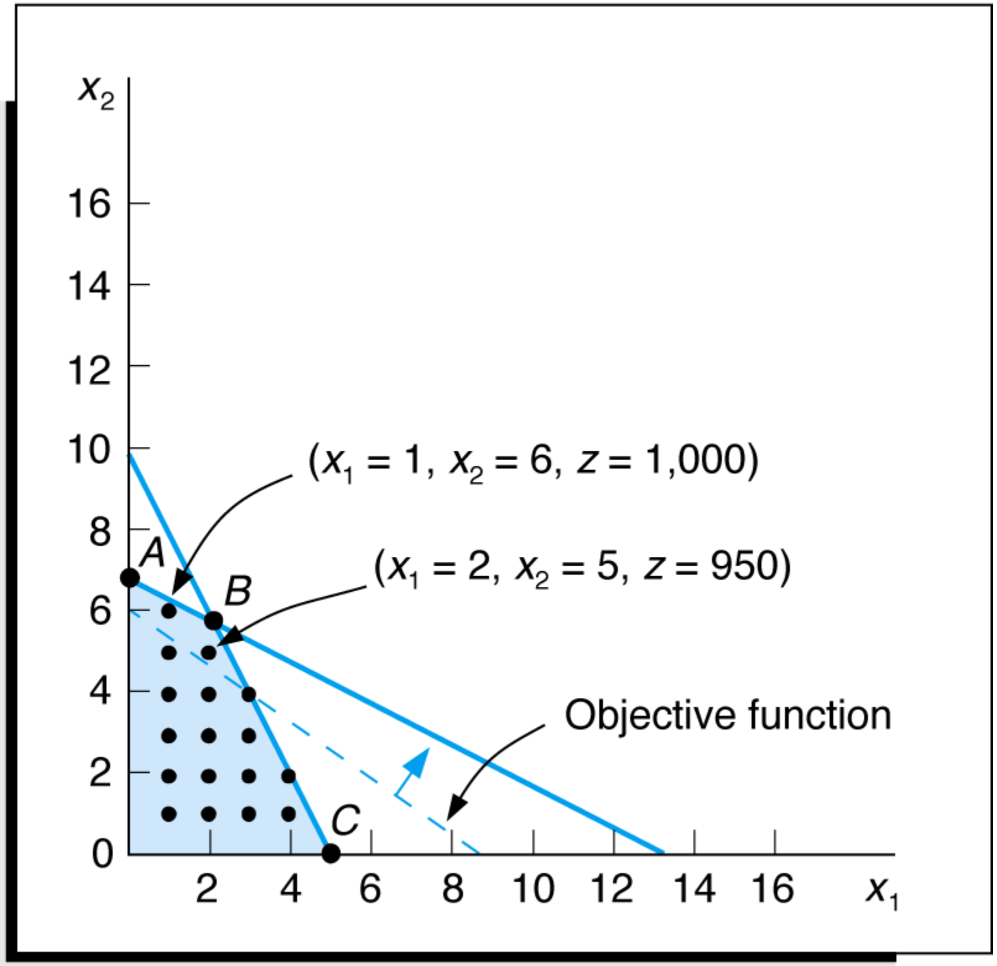
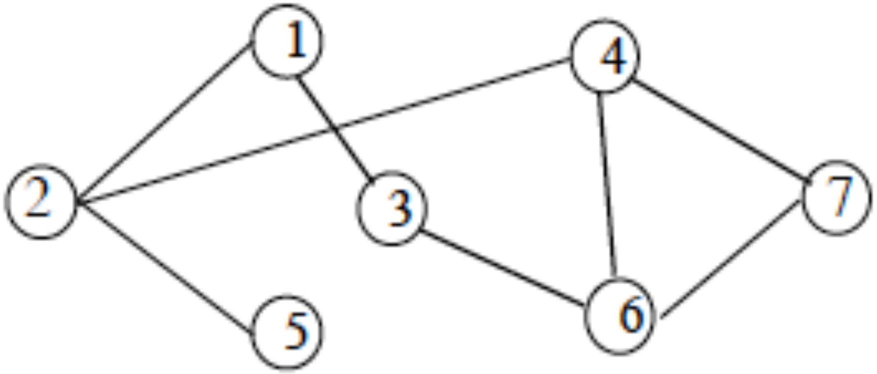
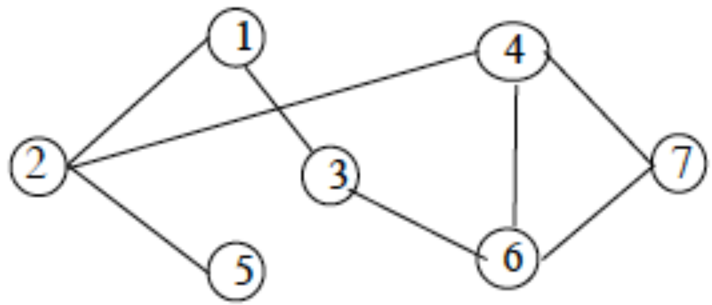


Figure 5.1 Feasible Solution Space with Integer Solution Points



- Find a maximum-size set of vertices that have no edges between any pair



$$v_i = \begin{cases} 1 \text{ if vertex i is in the MIS} \\ 0 \text{ otherwise} \end{cases}$$

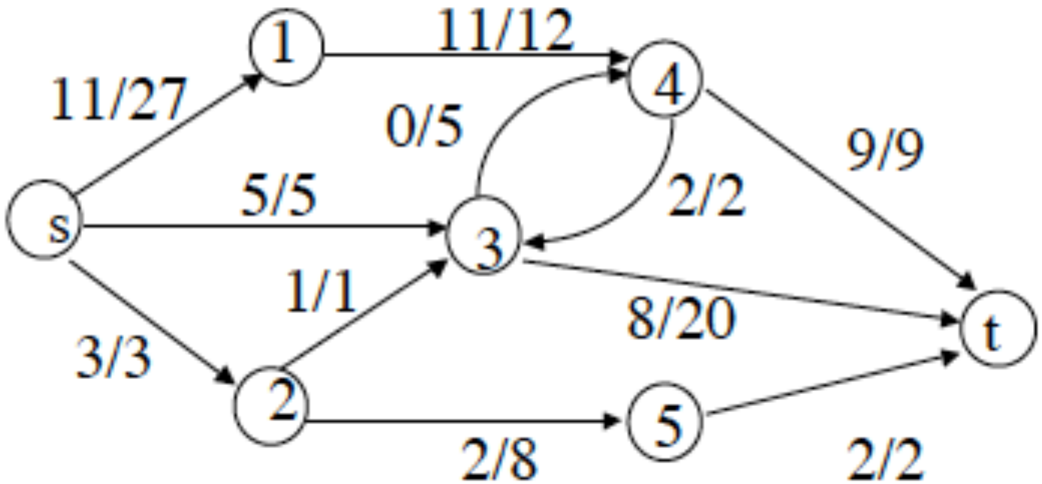
$$\max \sum v_i$$

$$\text{s.t. } v_i + v_j \leq 1 \quad \forall (i, j) \in E$$

$$v_i \in \{0,1\}$$

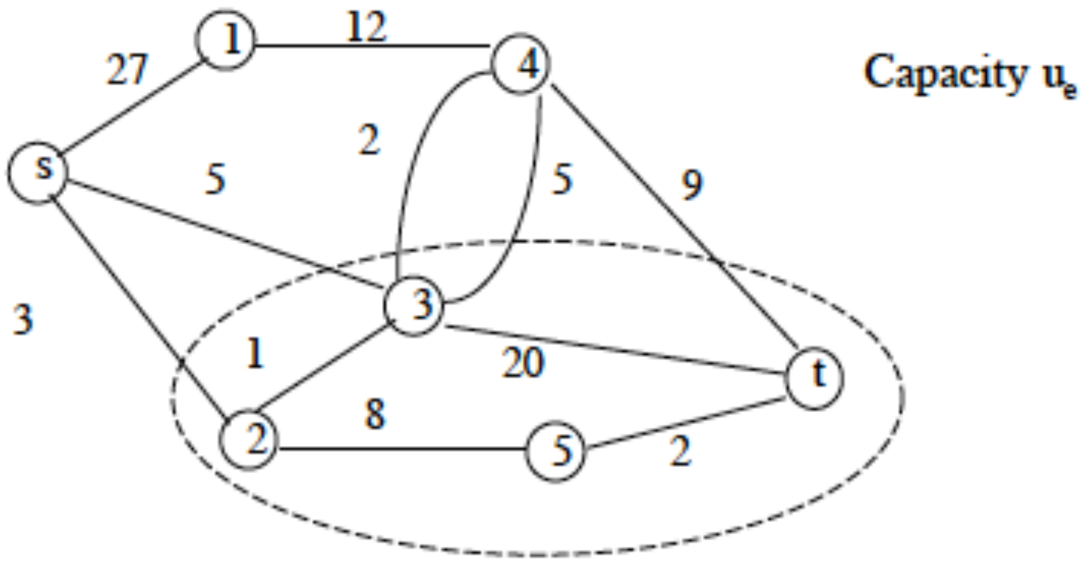
Project 4 – Network Flow Problem

Network Flow



- Source(s) s, sink (consumers) t
- Capacity (bottom number)
- Flow (top number)
- Maximize flow from s to t obeying
  - Capacity constraints on edges
  - Conservation constraints on all nodes other than s,t

Min Cut Problem



- Special nodes s and t
- Each edge e has capacity u_e. Set of edges S has capacity $\sum_{e \in S} u_e$
- Partition vertex set V into S,T where $s \in S$ and $t \in T$
- A cut is the edges (u,v) such that $u \in S$ and $v \in T$

Find a cut with minimum capacity

Algorithms

- Use IP to solve the network flow problem
- Use IP to solve the min-cut problem